SHORT FORMULA
MATHEMATICS

STRAIGHT LINE

1. Distance Formula: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

2. Section Formula : $x = \frac{mx_2 \pm nx_1}{m \pm n}$; $y = \frac{my_2 \pm ny_1}{m \pm n}$.

3. Centroid, Incentre & Excentre:

Centroid G $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$, Incentre I $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

Excentre $I_1\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$

4. Area of a Triangle:

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

5. Slope Formula:

(i) Line Joining two points $(x_1\ y_1)$ & $(x_2\ y_2)$, $m = \frac{y_1 - y_2}{x_1 - x_2}$

6. Condition of collinearity of three points: $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

7. Angle between two straight lines : $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

8. Two Lines : $ax + by + c = 0$ and $a'x + b'y + c' = 0$ two lines

1. parallel if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$.

2. Distance between two parallel lines = $\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.

3 Perpendicular : If $aa' + bb' = 0$.

9. A point and line:

1. Distance between point and line = $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$.

2. Reflection of a point about a line: $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = -2\,\dfrac{ax_1+by_1+c}{a^2+b^2}$

3. Foot of the perpendicular from a point on the line is $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = -\dfrac{ax_1+by_1+c}{a^2+b^2}$

10. Bisectors of the angles between two lines: $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = \pm\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

11. Condition of Concurrency :

of three straight lines $a_ix + b_iy + c_i = 0$, i = 1,2,3 is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

12. A Pair of straight lines through origin: $ax^2 + 2hxy + by^2 = 0$

If θ is the acute angle between the pair of straight lines, then $\tan\theta = \left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right|$.

CIRCLE

1. Intercepts made by Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the Axes:

(a) $2\sqrt{g^2-c}$ on x -axis (b) $2\sqrt{f^2-c}$ on y - aixs

2. Parametric Equations of a Circle: $x = h + r\cos\theta$; $y = k + r\sin\theta$

3. Tangent :

(a) Slope form : $y = mx \pm a\sqrt{1+m^2}$

(b) Point form : $xx_1 + yy_1 = a^2$ or T = o

(c) Parametric form : $x\cos\alpha + y\sin\alpha = a$.

4. Pair of Tangents from a Point: $SS_1 = T^2$.

5. Length of a Tangent : Length of tangent is $\sqrt{S_1}$

6. Director Circle: $x^2 + y^2 = 2a^2$ for $x^2 + y^2 = a^2$

7. Chord of Contact: T = 0

1. Length of chord of contact = $\dfrac{2LR}{\sqrt{R^2+L^2}}$

2. Area of the triangle formed by the pair of the tangents & its chord of contact = $\dfrac{RL^3}{R^2+L^2}$

3. Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\frac{2RL}{L^2 - R^2}\right)$

4. Equation of the circle circumscribing the triangle PT_1T_2 is : $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

8. **Condition of orthogonality of Two Circles:** $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

9. **Radical Axis :** $S_1 - S_2 = 0$ i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.

10. **Family of Circles:** $S_1 + KS_2 = 0$, $S + KL = 0$.

PARABOLA

1. **Equation of standard parabola :**
$y^2 = 4ax$, Vertex is (0, 0), focus is (a, 0), Directrix is $x + a = 0$ and Axis is $y = 0$
Length of the latus rectum = 4a, ends of the latus rectum are L(a, 2a) & L' (a, – 2a).

2. **Parametric Representation:** $x = at^2$ & $y = 2at$

3. **Tangents to the Parabola $y^2 = 4ax$:**

1. Slope form $y = mx + \frac{a}{m}$ $(m \neq 0)$ 2. Parametric form $ty = x + at^2$

3. Point form T = 0

4. **Normals to the parabola $y^2 = 4ax$:**

$y - y_1 = -\frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ; $y = mx - 2am - am^3$ at $(am^2, -2am)$; $y + tx = 2at + at^3$ at $(at^2, 2at)$.

ELLIPSE

1. **Standard Equation :** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ & $b^2 = a^2(1 - e^2)$.

Eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$, $(0 < e < 1)$, **Directrices :** $x = \pm\frac{a}{e}$.

Focii : $S \equiv (\pm ae, 0)$. Length of, major axes = 2a and minor axes = 2b
Vertices : $A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

Latus Rectum : = $\frac{2b^2}{a} = 2a(1 - e^2)$

2. **Auxiliary Circle :** $x^2 + y^2 = a^2$

3. **Parametric Representation :** $x = a\cos\theta$ & $y = b\sin\theta$

4. **Position of a Point w.r.t. an Ellipse:**

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > <$ or $= 0$.

5. **Line and an Ellipse:** The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

6. **Tangents:** Slope form: $y = mx \pm \sqrt{a^2m^2 + b^2}$, Point form : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$,

Parametric form: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

7. **Normals:** $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$, ax. sec θ – by cosec θ = $(a^2 - b^2)$, $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

8. **Director Circle:** $x^2 + y^2 = a^2 + b^2$

HYPERBOLA

1. **Standard Equation:** Standard equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$.

   Focii : S ≡ (± ae, 0) **Directrices :** $x = \pm \frac{a}{e}$

   Vertices : A ≡ (± a, 0)

   Latus Rectum (ℓ) : $\ell = \frac{2b^2}{a} = 2a(e^2 - 1)$.

2. **Conjugate Hyperbola :** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each.

3. **Auxiliary Circle :** $x^2 + y^2 = a^2$.

4. **Parametric Representation :** x = a sec θ & y = b tan θ

5. **Position of A Point 'P' w.r.t. A Hyperbola :**

   $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ >, = or < 0 according as the point (x_1, y_1) lies inside, on or outside the curve.

6. **Tangents :**

   (i) **Slope Form :** $y = mx \pm \sqrt{a^2m^2 - b^2}$

   (ii) **Point Form :** at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

   (iii) **Parametric Form :** $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

7. **Normals :**

   (a) at the point P (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$.

   (b) at the point P (a sec θ, b tan θ) is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

   (c) Equation of normals in terms of its slope 'm' are $y = mx \pm \frac{(a^2 + b^2)m}{\sqrt{a^2 - b^2m^2}}$.

8. **Asymptotes :** $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$. Pair of asymptotes : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

9. Rectangular Or Equilateral Hyperbola : $xy = c^2$, eccentricity is $\sqrt{2}$.

Vertices : $(\pm c, \pm c)$; Focii : $\left(\pm\sqrt{2}c, \pm\sqrt{2}c\right)$. Directrices : $x + y = \pm\sqrt{2}\,c$

Latus Rectum (l) : $\ell = 2\sqrt{2}\,c$ = T.A. = C.A.

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of the tangent at P (x_1, y_1) is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at P (t) is $\frac{x}{t} + t\,y = 2\,c$.

Equation of the normal at P (t) is $x\,t^3 - y\,t = c\,(t^4 - 1)$.

Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

LIMIT OF FUNCTION

1. Limit of a function f(x) is said to exist as $x \to a$ when,

$$\underset{h\to 0^+}{\text{Limit}}\ f(a-h) = \underset{h\to 0^+}{\text{Limit}}\ f(a+h) = \text{some finite value M.}$$

(Left hand limit) (Right hand limit)

2. Indeterminant Forms:

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^0, 0^0$, and 1^∞.

3. Standard Limits:

$$\underset{x\to 0}{\text{Limit}}\ \frac{\sin x}{x} = \underset{x\to 0}{\text{Limit}}\ \frac{\tan x}{x} = \underset{x\to 0}{\text{Limit}}\ \frac{\tan^{-1}x}{x} = \underset{x\to 0}{\text{Limit}}\ \frac{\sin^{-1}x}{x} = \underset{x\to 0}{\text{Limit}}\ \frac{e^x - 1}{x} = \underset{x\to 0}{\text{Limit}}\ \frac{\ell n(1+x)}{x} = 1$$

$$\underset{x\to 0}{\text{Limit}}\ (1+x)^{1/x} = \underset{x\to\infty}{\text{Limit}}\left(1+\frac{1}{x}\right)^x = e,\quad \underset{x\to 0}{\text{Limit}}\ \frac{a^x - 1}{x} = \log_e a,\ a > 0,\quad \underset{x\to a}{\text{Limit}}\ \frac{x^n - a^n}{x - a} = na^{n-1}.$$

4. Limits Using Expansion

(i) $a^x = 1 + \frac{x\ln a}{1!} + \frac{x^2\ln^2 a}{2!} + \frac{x^3\ln^3 a}{3!} + \ldots\ldots\ldots a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \text{for } -1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$

(vii) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots.$

(viii) $\sin^{-1}x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots\ldots$

(x) for $|x| < 1$, $n \in R$ $(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\ldots\ldots\ldots\infty$

5. Limits of form 1^∞, 0^0, ∞^0

Also for $(1)^\infty$ type of problems we can use following rules.

$\lim_{x\to 0} (1 + x)^{1/x} = e$, $\lim_{x\to a} [f(x)]^{g(x)}$, where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a = \lim_{x\to a} = e^{\lim_{x\to a}[f(x)-1]\, g(x)}$

6. Sandwich Theorem or Squeeze Play Theorem:

If $f(x) \le g(x) \le h(x)\ \forall x$ & $\underset{x\to a}{\text{Limit}}\ f(x) = \ell = \underset{x\to a}{\text{Limit}}\ h(x)$ then $\underset{x\to a}{\text{Limit}}\ g(x) = \ell$.

METHOD OF DIFFERENTIATION

1. Differentiation of some elementary functions

1. $\frac{d}{dx}(x^n) = nx^{n-1}$ **2.** $\frac{d}{dx}(a^x) = a^x \ell n\, a$ **3.** $\frac{d}{dx}(\ell n|x|) = \frac{1}{x}$ **4.** $\frac{d}{dx}(\log_a x) = \frac{1}{x\, \ell n\, a}$

5. $\frac{d}{dx}(\sin x) = \cos x$ **6.** $\frac{d}{dx}(\cos x) = -\sin x$ **7.** $\frac{d}{dx}(\sec x) = \sec x\ \tan x$

8. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$ **9.** $\frac{d}{dx}(\tan x) = \sec^2 x$ **10.** $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

2. Basic Theorems

1. $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$ **2.** $\frac{d}{dx}(k\, f(x)) = k \frac{d}{dx} f(x)$ **3.** $\frac{d}{dx}(f(x) \,.\, g(x)) = f(x)\, g'(x) + g(x)\, f'(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\, f'(x) - f(x) g'(x)}{g^2(x)}$ **5.** $\frac{d}{dx}(f(g(x))) = f'(g(x))\, g'(x)$

Derivative Of Inverse Trigonometric Functions.

$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}$, $\frac{d\cos^{-1}x}{dx} = -\frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$.

$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}$, $\frac{d\cot^{-1}x}{dx} = -\frac{1}{1+x^2}$ $(x \in R)$

$\frac{d\sec^{-1}x}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$, $\frac{d\,\text{cosec}^{-1}x}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}$, for $x \in (-\infty, -1) \cup (1, \infty)$

3. Differentiation using substitution

Following substitutions are normally used to sumplify these expression.

(i) $\sqrt{x^2 + a^2}$ by substituting $x = a \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(ii) $\sqrt{a^2 - x^2}$ by substituting $x = a \sin \theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(iii) $\sqrt{x^2 - a^2}$ by substituting $x = a \sec\theta$, where $\theta \in [0, \pi]$, $\theta \neq \frac{\pi}{2}$

(iv) $\sqrt{\frac{x+a}{a-x}}$ by substituting $x = a \cos\theta$, where $\theta \in (0, \pi]$.

4. Parametric Differentiation

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

5. Derivative of one function with respect to another

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

6. If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

APPLICATION OF DERIVATIVES

1. Equation of tangent and normal

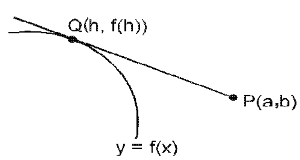
Tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

And normal at (x_1, y_1) is $(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

2. Tangent from an external point

Given a point P(a, b) which does not lie on the curve y = f(x), then the equation of possible tangents to the curve y = f(x), passing through (a, b) can be found by solving for the point of contact Q.

$$f'(h) = \frac{f(h) - b}{h - a}$$



Figure

And equation of tangent is $y - b = \frac{f(h) - b}{h - a}(x - a)$

3. Length of tangent, normal, subtangent, subnormal

(i) $PT = |k|\sqrt{1 + \frac{1}{m^2}}$ = Length of Tangent

(ii) $PN = |k|\sqrt{1 + m^2}$ = Length of Normal

(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent

(iv) $MN = |km|$ = Length of subnormal.

4. Angle between the curves

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves (as shown in figure).

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

5. Shortest distance between two curves

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

6. Rolle's Theorem :

If a function f defined on [a, b] is

(i) continuous on [a, b] (ii) derivable on (a, b) and

(iii) f(a) = f(b),

then there exists at least one real number c between a and b ($a < c < b$) such that $f'(c) = 0$

7. Lagrange's Mean Value Theorem (LMVT) :

If a function f defined on [a, b] is

(i) continuous on [a, b] and (ii) derivable on (a, b)

then there exists at least one real numbers between a and b ($a < c < b$) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

8. Useful Formulae of Mensuration to Remember :

1. Volume of a cuboid = ℓbh.
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3
4. Surface area of cube = $6a^2$
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height)
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.
12. Volume of a prism = (area of the base) × (height).
13. Lateral surface area of a prism = (perimeter of the base) × (height).
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base)
    (Note that lateral surfaces of a prism are all rectangle).

15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).

16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
(Note that slant surfaces of a pyramid are triangles).

INDEFINITE INTEGRATION

1. If f & g are functions of x such that $g'(x) = f(x)$ then,

$\int f(x)\,dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x)+c\} = f(x)$, where c is called the **constant of integration**.

2. Standard Formula:

(i) $\int (ax + b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\,\ell n\,(ax + b) + c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + c$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\,\frac{a^{px+q}}{\ell n a} + c;\ a > 0$

(v) $\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + c$

(vi) $\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b) + c$

(vii) $\int \tan(ax + b)\,dx = \frac{1}{a}\,\ell n \sec(ax + b) + c$

(viii) $\int \cot(ax + b)\,dx = \frac{1}{a}\,\ell n \sin(ax + b) + c$

(ix) $\int \sec^2(ax + b)\,dx = \frac{1}{a}\tan(ax + b) + c$

(x) $\int \text{cosec}^2(ax + b)\,dx = -\frac{1}{a}\cot(ax + b) + c$

(xiii) $\int \sec x\,dx = \ell n\,(\sec x + \tan x) + c$ **OR** $\ell n \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(xiv) $\int \text{cosec}\,x\,dx = \ell n\,(\text{cosec}x - \cot x) + c$ **OR** $\ell n \tan\frac{x}{2} + c$ **OR** $-\ell n\,(\text{cosec}x + \cot x) + c$

(xv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

(xvi) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xvii) $\int \frac{dx}{|x|\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xviii) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ell n\left[x + \sqrt{x^2 + a^2}\right] + c$

(xix) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ell n\left[x + \sqrt{x^2 - a^2}\right] + c$

(xx) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\,\ell n\left|\frac{a+x}{a-x}\right| + c$

(xxi) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\,\ell n\left|\frac{x-a}{x+a}\right| + c$

(xxii) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxiii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\,\ell n\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c$

(xxiv) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\,\ell n\left(\frac{x+\sqrt{x^2-a^2}}{a}\right) + c$

3. Integration by Subsitutions

If we subsitute f(x) = t, then f ′(x) dx = dt

4. Integration by Part :

$$\int (f(x)\,g(x))\,dx = f(x)\int (g(x))dx - \int\left(\frac{d}{dx}(f(x))\int (g(x))\,dx\right)dx$$

5. Integration of type $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$

Make the substitution $x + \frac{b}{2a} = t$

6. Integration of type

$$\int \frac{px+q}{ax^2+bx+c}\,dx,\ \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx,\ \int (px+q)\sqrt{ax^2+bx+c}\,dx$$

Make the substitution $x + \frac{b}{2a} = t$, then split the integral as some of two integrals one containing the linear term and the other containing constant term.

7. Integration of trigonometric functions

(i) $\int \frac{dx}{a+b\sin^2 x}$ **OR** $\int \frac{dx}{a+b\cos^2 x}$ **OR** $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$ **put tan x = t.**

(ii) $\int \frac{dx}{a+b\sin x}$ **OR** $\int \frac{dx}{a+b\cos x}$ **OR** $\int \frac{dx}{a+b\sin x + c\cos x}$ **put** $\tan\frac{x}{2} = t$

(iii) $\int \frac{a.\cos x + b.\sin x + c}{\ell.\cos x + m.\sin x + n}$ **dx. Express Nr** $\equiv$ **A(Dr) + B** $\frac{d}{dx}$ **(Dr) + c & proceed.**

8. Integration of type

$\int \frac{x^2 \pm 1}{x^4 + Kx^2 + 1}$ **dx where K is any constant.**

Divide Nr & Dr by x² & put $x \mp \frac{1}{x} = t$.

9. Integration of type

$\int \frac{dx}{(ax+b)\sqrt{px+q}}$ **OR** $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ **; put px + q = t^2.**

10. Integration of type

$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, **put ax + b =** $\frac{1}{t}$; $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$, **put x =** $\frac{1}{t}$

DEFINITE INTEGRATION

Properties of definite integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ **2.** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ **3.** $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x)+f(-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(-x)=f(x) \\ 0, & f(-x)=-f(x) \end{cases}$

5. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ **6.** $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

7. $\int_0^{2a} f(x)\,dx = \int_0^a (f(x)+f(2a-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(2a-x)=f(x) \\ 0, & f(2a-x)=-f(x) \end{cases}$

8. If f(x) is a periodic function with period T, then

$\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in z,$ $\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in z,\ a \in R$

$\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx,\ m, n \in z,$ $\int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)\,dx,\ n \in z,\ a \in R$

$\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx,\ n \in z,\ a, b \in R$

9. If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then $\int_a^b \psi(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

10. If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

11. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$ **12.** If $f(x) \ge 0$ on [a, b] then $\int_a^b f(x)\,dx \ge 0$

Leibnitz Theorem : If $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$, then $\frac{dF(x)}{dx} = h'(x)\,f(h(x)) - g'(x)\,f(g(x))$

BASICS

Intervals :

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers a, b $\in$ R such that a < b, we can define four types of intervals as follows :

		Symbols Used
(i)	Open interval : (a, b) = $\{x : a < x < b\}$ i.e. end points are not included.	( ) or] [
(ii)	Closed interval : [a, b] = $\{x : a \le x \le b\}$ i.e. end points are also included. This is possible only when both a and b are finite.	[ ]
(iii)	Open-closed interval : (a, b] = $\{x : a < x \le b\}$	( ] or]]
(iv)	Closed - open interval : [a, b) = $x : a \le x < b\}$	[ ) or [[

The infinite intervals are defined as follows :

(i) $(a, \infty) = \{x : x > a\}$

(ii) $[a, \infty) = \{x : x \ge a\}$

(iii) $(-\infty, b) = \{x : x < b\}$

(iv) $(\infty, b] = \{x : x \le b\}$

(v) $(-\infty, \infty) = \{x : x \in R\}$

Properties of Modulus :

For any a, b $\in$ R

$|a| \ge 0$, $|a| = |-a|$, $|a| \ge a, |a| \ge -a$, $|ab| = |a|\,|b|$, $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$,

$|a + b| \le |a| + |b|$, $|a - b| \ge ||a| - |b||$

Trigonometric Functions of Sum or Difference of Two Angles:

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\therefore$ $2\sin A \cos B = \sin(A+B) + \sin(A-B)$ and and $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\therefore$ $2\cos A \cos B = \cos(A+B) + \cos(A-B)$ and $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$

(e) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(f) $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.

Factorisation of the Sum or Difference of Two Sines or Cosines:

(a) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(b) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(d) $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

Multiple and Sub-multiple Angles :

(a) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$; $2\cos^2\dfrac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\dfrac{\theta}{2} = 1 - \cos\theta$.

(c) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$, $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$ (d) $\sin 3A = 3\sin A - 4\sin^3 A$

(e) $\cos 3A = 4\cos^3 A - 3\cos A$ (f) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

Important Trigonometric Ratios:

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)$; $\tan n\pi = 0$, where $n \in I$

(b) $\sin 15°$ or $\sin \dfrac{\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos \dfrac{5\pi}{12}$;

$\cos 15°$ or $\cos \dfrac{\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin \dfrac{5\pi}{12}$;

$\tan 15° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

(c) $\sin \dfrac{\pi}{10}$ or $\sin 18° = \dfrac{\sqrt{5}-1}{4}$ & $\cos 36°$ or $\cos \dfrac{\pi}{5} = \dfrac{\sqrt{5}+1}{4}$

Range of Trigonometric Expression:

$$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$$

Sine and Cosine Series :

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\left(\alpha + \overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos\left(\alpha + \overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

Trigonometric Equations

Principal Solutions: Solutions which lie in the interval $[0, 2\pi)$ are called **Principal solutions.**

General Solution :

(i) $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

(ii) $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

(iii) $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$, $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

QUADRATIC EQUATIONS

1. Quadratic Equation : $ax^2 + bx + c = 0$, $a \neq 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, The expression $b^2 - 4ac \equiv D$ is called discriminant of quadratic equation.

If α, β are the roots, then (a) $\alpha + \beta = -\dfrac{b}{a}$ (b) $\alpha\beta = \dfrac{c}{a}$

A quadratic equation whose roots are α & β, is $(x - \alpha)(x - \beta) = 0$ i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

2. Nature of Roots:

Consider the quadratic equation, $ax^2 + bx + c = 0$ having α, β as its roots; $D \equiv b^2 - 4ac$

- $D = 0$
  Roots are equal $\alpha = \beta = -b/2a$
- $D \neq 0$
  Roots are unequal
  - $a, b, c \in R$ & $D > 0$
    Roots are real
    - $a, b, c \in Q$ &
      D is a perfect square
      $\Rightarrow$ Roots are rational
      $\downarrow$
      $a = 1$, $b, c \in I$ & D is a perfect square
      $\Rightarrow$ Roots are integral.
    - $a, b, c \in Q$ &
      D is not a perfect square
      $\Rightarrow$ Roots are irrational
      i.e. $\alpha = p + \sqrt{q}$, $\beta = p - \sqrt{q}$
  - $a, b, c \in R$ & $D < 0$
    Roots are imaginary $\alpha = p + iq$, $\beta = p - iq$

3. Common Roots:

Consider two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ & $a_2x^2 + b_2x + c_2 = 0$.

(i) If two quadratic equations have both roots common, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

(ii) If only one root α is common, then $\alpha = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} = \dfrac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$

4. Range of Quadratic Expression f (x) = $ax^2 + bx + c$.

Range in restricted domain: Given $x \in [x_1, x_2]$

(a) If $-\dfrac{b}{2a} \notin [x_1, x_2]$ then, $f(x) \in \left[\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}\right]$

(b) If $-\dfrac{b}{2a} \in [x_1, x_2]$ then, $f(x) \in \left[\min\left\{f(x_1), f(x_2), -\dfrac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\dfrac{D}{4a}\right\}\right]$

5. Location of Roots:

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

(i) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'x_0' are $b^2 - 4ac \geq 0$; $f(x_0) > 0$ & $(-b/2a) > x_0$.

(ii) Conditions for both the roots of $f(x) = 0$ to be smaller than a specified number 'x_0' are $b^2 - 4ac \geq 0$; $f(x_0) > 0$ & $(-b/2a) < x_0$.

(iii) Conditions for both roots of $f(x) = 0$ to lie on either side of the number 'x_0' (in other words the number 'x_0' lies between the roots of $f(x) = 0$), is $f(x_0) < 0$.

(iv) Conditions that both roots of $f(x) = 0$ to be confined between the numbers x_1 and x_2, $(x_1 < x_2)$ are $b^2 - 4ac \geq 0$; $f(x_1) > 0$; $f(x_2) > 0$ & $x_1 < (-b/2a) < x_2$.

(v) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (x_1, x_2) i.e. $x_1 < x < x_2$ is $f(x_1).\, f(x_2) < 0$.

SEQUENCE & SERIES

An arithmetic progression (A.P.) : $a, a + d, a + 2d, \ldots\ldots.\ a + (n - 1)d$ is an A.P.

Let a be the first term and d be the common difference of an A.P., then n^{th} term $= t_n = a + (n - 1)d$

The sum of first n terms of are A.P.

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + \ell]$$

r^{th} term of an A.P. when sum of first r terms is given is $t_r = S_r - S_{r-1}$.

Properties of A.P.

(i) If a, b, c are in A.P. $\Rightarrow 2b = a + c$ & if a, b, c, d are in A.P. $\Rightarrow a + d = b + c$.

(ii) Three numbers in A.P. can be taken as $a - d, a, a + d$; four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$; five numbers in A.P. are $a - 2d, a - d, a, a + d, a + 2d$ & six terms in A.P. are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iii) Sum of the terms of an A.P. equidistant from the beginning & end = sum of first & last term.

Arithmetic Mean (Mean or Average) (A.M.):

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

n – Arithmetic Means Between Two Numbers:

If a, b are any two given numbers & $a, A_1, A_2, \ldots, A_n, b$ are in A.P. then $A_1, A_2, \ldots A_n$ are the n A.M.'s between a & b. $A_1 = a + \frac{b-a}{n+1}$, $A_2 = a + \frac{2(b-a)}{n+1}$, ..., $A_n = a + \frac{n(b-a)}{n+1}$

$\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

Geometric Progression: $a, ar, ar^2, ar^3, ar^4, \ldots\ldots$ is a G.P. with a as the first term & r as common ratio.

(i) n^{th} term $= a r^{n-1}$ (ii) Sum of the first n terms i.e. $S_n = \begin{cases} \frac{a(r^n - 1)}{r - 1} & , r \neq 1 \\ na & , r = 1 \end{cases}$

(iii) Sum of an infinite G.P. when $|r| < 1$ is given by $S_\infty = \frac{a}{1-r}\ (|r| < 1)$.

Geometric Means (Mean Proportional) (G.M.):

If a, b, c > 0 are in G.P., b is the G.M. between a & c, then $b^2 = ac$

n–Geometric Means Between positive number a, b: If a, b are two given numbers & a, $G_1, G_2, \ldots, G_n$, b are in G.P.. Then $G_1, G_2, G_3, \ldots, G_n$ are n G.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}, \ldots, G_n = a(b/a)^{n/n+1}$

Harmonic Mean (H.M.):

If a, b, c are in H.P., b is the H.M. between a & c, then $b = \frac{2ac}{a+c}$.

H.M. H of $a_1, a_2, \ldots, a_n$ is given by $\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1}+\frac{1}{a_2}+\ldots\ldots+\frac{1}{a_n}\right]$

Relation between means :

$G^2 = AH$, A.M. $\geq$ G.M. $\geq$ H.M. and A.M. = G.M. = H.M. if $a_1 = a_2 = a_3 = \ldots\ldots\ldots = a_n$

Important Results

(i) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$. (ii) $\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r$. (iii) $\sum_{r=1}^{n} k = nk$; where k is a constant.

(iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots\ldots\ldots + n = \frac{n(n+1)}{2}$ (v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$

(vii) $2\sum_{i<j=1}^{n} a_i a_j = (a_1 + a_2 + \ldots\ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots\ldots + a_n^2)$

BINOMIAL THEOREM

1. Statement of Binomial theorem : If a, b $\in$ R and n $\in$ N, then

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, a^0 b^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$$

2. Properties of Binomial Theorem :

(i) **General term :** $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

(ii) **Middle term (s) :**

(a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)$th term.

(b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2}+1\right)$th terms.

3. Multinomial Theorem : $(x_1 + x_2 + x_3 + \ldots\ldots\ldots x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \frac{n!}{r_1!\,r_2!\ldots r_k!}\, x_1^{r_1} . x_2^{r_2} \ldots x_k^{r_k}$

Here total number of terms in the expansion = ${}^{n+k-1}C_{k-1}$

4. Application of Binomial Theorem :

If $(\sqrt{A}+B)^n = I + f$ where I and n are positive integers, n being odd and $0 < f < 1$ then $(I + f) f = k^n$ where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$.
If n is an even integer, then $(I + f)(1 - f) = k^n$

5. Properties of Binomial Coefficients :

(i) $^nC_0 + {^nC_1} + {^nC_2} + \ldots\ldots + {^nC_n} = 2^n$
(ii) $^nC_0 - {^nC_1} + {^nC_2} - {^nC_3} + \ldots\ldots + (-1)^n\, {^nC_n} = 0$
(iii) $^nC_0 + {^nC_2} + {^nC_4} + \ldots = {^nC_1} + {^nC_3} + {^nC_5} + \ldots = 2^{n-1}$
(iv) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$ (v) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

6. Binomial Theorem For Negative Integer Or Fractional Indices

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}x^r + \ldots, |x| < 1.$$

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}x^r$$

PERMUTATION & COMBINNATION

1. Arrangement : number of permutations of n different things taken r at a time =

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

2. Circular Permutation :

The number of circular permutations of n different things taken all at a time is; (n – 1)!

3. Selection : Number of combinations of n different things taken r at a time = $^nC_r = \frac{n!}{r!\,(n-r)!} = \frac{^nP_r}{r!}$

4. The number of permutations of 'n' things, taken all at a time, when 'p' of them are similar & of one type, q of them are similar & of another type, 'r' of them are similar & of a third type & the remaining n – (p + q + r) are all different is $\frac{n!}{p!\,q!\,r!}$.

5. Selection of one or more objects

(a) Number of ways in which atleast one object be selected out of 'n' distinct objects is
$^nC_1 + {^nC_2} + {^nC_3} + \ldots\ldots + {^nC_n} = 2^n - 1$

(b) Number of ways in which atleast one object may be selected out of 'p' alike objects of one type 'q' alike objects of second type and 'r' alike of third type is
$(p + 1)(q + 1)(r + 1) - 1$

(c) Number of ways in which atleast one object may be selected from 'n' objects where 'p' alike of one type 'q' alike of second type and 'r' alike of third type and rest
n – (p + q + r) are different, is
$(p + 1)(q + 1)(r + 1)\, 2^{n-(p+q+r)} - 1$

6. Multinomial Theorem :

Coefficient of x^r in expansion of $(1 - x)^{-n} = {^{n+r-1}C_r}$ $(n \in N)$

7. Let $N = p^a\, q^b\, r^c \ldots\ldots$ where p, q, r...... are distinct primes & a, b, c..... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is = (a + 1) (b + 1) (c + 1)........

(b) The sum of these divisors is =
$(p^0 + p^1 + p^2 + ... + p^a)\ (q^0 + q^1 + q^2 + ... + q^b)\ (r^0 + r^1 + r^2 + ... + r^c)........$

(c) Number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \frac{1}{2}(a+1)(b+1)(c+1).... & \text{if N is not a perfect square} \\ \frac{1}{2}[(a+1)(b+1)(c+1)....+1] & \text{if N is a perfect square} \end{cases}$$

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

8. Dearrangement :

Number of ways in which 'n' letters can be put in 'n' corresponding envelopes such that no letter goes to correct envelope is $n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}...........+(-1)^n\frac{1}{n!}\right)$

PROBABILITY

1. Classical (A priori) Definition of Probability :

If an experiment results in a total of (m + n) outcomes which are equally likely and mutually exclusive with one another and if 'm' outcomes are favorable to an event 'A' while 'n' are unfavorable, then the probability of occurrence of the event 'A' = $P(A) = \frac{m}{m+n} = \frac{n(A)}{n(S)}$.

We say that odds in favour of 'A' are m : n, while odds against 'A' are n : m.

$$P(\bar{A}) = \frac{n}{m+n} = 1 - P(A)$$

2. Addition theorem of probability : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

De Morgan's Laws : (a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cap B)^c = A^c \cup B^c$

Distributive Laws : (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(i) $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(ii) P (at least two of A, B, C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(iii) P(exactly two of A, B, C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(iv) P(exactly one of A, B, C occur) =
$P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

3. Conditional Probability : $P(A/B) = \frac{P(A \cap B)}{P(B)}$.

4. Binomial Probability Theorem

If an experiment is such that the probability of success or failure does not change with trials, then the probability of getting exactly r success in n trials of an experiment is ${}^nC_r\, p^r q^{n-r}$, where 'p' is the probability of a success and q is the probability of a failure. Note that p + q = 1.

5. Expectation :

If a value M_i is associated with a probability of p_i, then the expectation is given by $\sum p_i M_i$.

6. Total Probability Theorem : $P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)$

7. Bayes' Theorem :

If an event A can occur with one of the n mutually exclusive and exhaustive events B_1, B_2, ..., B_n and the probabilities $P(A/B_1)$, $P(A/B_2)$ .... $P(A/B_n)$ are known, then $P(B_i / A) = \dfrac{P(B_i)\,.\,P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\,.\,P(A/B_i)}$

$B_1, B_2, B_3, \ldots\ldots, B_n$

$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots\ldots \cup (A \cap B_n)$

$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots\ldots + P(A \cap B_n) = \sum_{i=1}^{n} P(A \cap B_i)$

8. Binomial Probability Distribution :

(i) Mean of any probability distribution of a random variable is given by : $\mu = \dfrac{\Sigma p_i x_i}{\Sigma p_i} = \Sigma p_i x_i$

(ii) Variance of a random variable is given by, $\sigma^2 = \Sigma (x_i - \mu)^2 \,.\, p_i = \Sigma p_i x_i^2 - \mu^2$

COMPLEX NUMBER

1. The complex number system

$z = a + ib$, then $a - ib$ is called congugate of z and is denoted by $\bar{z}$.

2. Equality In Complex Number: $z_1 = z_2 \Rightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $I_m(z_1) = I_m(z_2)$.

3. Representation Of A Complex Number:

4. Properties of arguments

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2m\pi$ for some integer m.

(ii) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) + 2m\pi$ for some integer m.

(iii) $\arg(z^2) = 2\arg(z) + 2m\pi$ for some integer m.

(iv) $\arg(z) = 0 \Leftrightarrow$ z is a positive real number

(v) $\arg(z) = \pm \pi/2 \Leftrightarrow$ z is purely imaginary and $z \neq 0$

5. Properties of conjugate

(i) $|z| = |\bar{z}|$ (ii) $z\bar{z} = |z|^2$ (iii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iv) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (v) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (vi) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ $(z_2 \neq 0)$

(vii) $|z_1 + z_2|^2 = (z_1 + z_2)\,\overline{(z_1 + z_2)} = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

(viii) $\overline{(\bar{z}_1)} = z$ (ix) If $w = f(z)$, then $\bar{w} = f(\bar{z})$

(x) $\arg(z) + \arg(\bar{z})$

6. Rotation theorem

If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then $\left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = \left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$

7. Demoivre's Theorem :

Case I : If n is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ .....(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + ...\ \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + ...\ + \theta_n)$

Case II : If p, q $\in$ Z and q $\neq$ 0 then $(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$

where k = 0, 1, 2, 3, ..., q – 1

8. Cube Root Of Unity :

(i) The cube roots of unity are 1, $\frac{-1 + i\sqrt{3}}{2}$, $\frac{-1 - i\sqrt{3}}{2}$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where r $\in$ I but is not the multiple of 3.

9. Logarithm Of A Complex Quantity :

(i) $\text{Log}_e(\alpha + i\beta) = \frac{1}{2}\text{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$ where n $\in$ I.

10. Geometrical Properties:

Distance formula : $|z_1 - z_2|$.

Section formula : $z = \frac{mz_2 + nz_1}{m+n}$ (internal division), $z = \frac{mz_2 - nz_1}{m-n}$ (external division)

(1) amp(z) = θ is a ray emanating from the origin inclined at an angle θ to the x– axis.

(2) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(3) The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t(z_1 - z_2)$ where t is a real parameter.

(4) The equation of circle having centre z_0 & radius ρ is :
$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form
$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + k = 0$, k is real. Centre is $-\alpha$ & radius $= \sqrt{\alpha\bar{\alpha} - k}$.
Circle will be real if $\alpha\bar{\alpha} - k \geq 0$..

(5) If $|z - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 & z_2

(6) If $\left|\frac{z - z_1}{z - z_2}\right| = k \neq 1, 0$, then locus of z is circle.

(7) If $\big||z - z_1| - |z - z_2|\big| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 & z_2.

VECTORS

1. Position Vector Of A Point:

let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then, $\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

DISTANCE FORMULA : Distance between the two points A $(\vec{a})$ and B $(\vec{b})$ is AB = $|\vec{a}-\vec{b}|$

SECTION FORMULA : $\vec{r}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$. Mid point of AB = $\dfrac{\vec{a}+\vec{b}}{2}$.

2. **Scalar Product Of Two Vectors:** $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$, where $|\vec{a}|$, $|\vec{b}|$ are magnitude of $\vec{a}$ and $\vec{b}$ respectively and θ is angle between $\vec{a}$ and $\vec{b}$.

1. i.i = j.j = k.k = 1; i.j = j.k = k.i = 0 ☞ projection of $\vec{a}$ on $\vec{b}=\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

2. If $\vec{a}=a_1i+a_2j+a_3k$ & $\vec{b}=b_1i+b_2j+b_3k$ then $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$

3. The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi=\dfrac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$, $0\le\phi\le\pi$

4. $\vec{a}.\vec{b}=0\Leftrightarrow\vec{a}\perp\vec{b}$ $(\vec{a}\neq0\ \vec{b}\neq0)$

3. **Vector Product Of Two Vectors:**

1. If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\ \vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed screw system.

2. Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.

3. $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$; $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$, $\hat{k}\times\hat{i}=\hat{j}$

4. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$

5. $\vec{a}\times\vec{b}=\vec{o}\Leftrightarrow\vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a}\neq0\ ,\ \vec{b}\neq0)$ i.e. $\vec{a}=K\vec{b}$, where K is a scalar.

6. Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n}=\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

☞ If $\vec{a},\vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC = $\dfrac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right]$. The points A, B & C are collinear if $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$

☞ Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$

☞ Lagrange's Identity : $(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2=\begin{vmatrix}\vec{a}.\vec{a}&\vec{a}.\vec{b}\\\vec{a}.\vec{b}&\vec{b}.\vec{b}\end{vmatrix}$

4. Scalar Triple Product:

- The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ is defined as: $\vec{a} \times \vec{b} . \vec{c} = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi$.
- Volume of tetrahydron $V = [\vec{a}\vec{b}\vec{c}]$
- In a scalar triple product the position of dot & cross can be interchanged i.e.

  $\vec{a} . (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) . \vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$
- $\vec{a} . (\vec{b} \times \vec{c}) = -\vec{a} . (\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$
- If $\vec{a} = a_1i + a_2j + a_3k$; $\vec{b} = b_1i + b_2j + b_3k$ & $\vec{c} = c_1i + c_2j + c_3k$ then $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

  In general, if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$ & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

  then $\left[\vec{a}\vec{b}\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \left[\vec{l}\,\vec{m}\,\vec{n}\right]$; where $\vec{\ell}$, $\vec{m}$ & $\vec{n}$ are non coplanar vectors.
- If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.
- Volume of tetrahedron OABC with O as origin & A($\vec{a}$), B($\vec{b}$) and C($\vec{c}$) be the vertices = $\left| \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}] \right|$
- The positon vector of the centroid of a tetrahedron if the pv's of its vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are given by

  $\frac{1}{4}\ [\vec{a} + \vec{b} + \vec{c} + \vec{d}]$.

5. Vector Triple Product: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} . \vec{c})\vec{b} - (\vec{a} . \vec{b})\vec{c}$, $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} . \vec{c})\vec{b} - (\vec{b} . \vec{c})\vec{a}$

- $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$, in general

6. Reciprocal System Of Vectors:

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$ then the two systems are called Reciprocal System of vectors, where $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

3-DIMENSION

1. **Vector representation of a point :** Position vector of point P (x, y, z) is $x\hat{i} + y\hat{j} + z\hat{k}$.
2. **Distance formula :** $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$, AB = $|\overrightarrow{OB} - \overrightarrow{OA}|$
3. **Distance of P from coordinate axes :** PA = $\sqrt{y^2 + z^2}$, PB = $\sqrt{z^2 + x^2}$, PC = $\sqrt{x^2 + y^2}$

4. Section Formula : $x = \frac{mx_2 + nx_1}{m+n}$, $y = \frac{my_2 + ny_1}{m+n}$, $z = \frac{mz_2 + nz_1}{m+n}$

Mid point : $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$, $z = \frac{z_1 + z_2}{2}$

5. Direction Cosines And Direction Ratios

(i) Direction cosines: Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ, m, n). Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) Direction ratios: Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\ell = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

(vi) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$, $m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$

6. Angle Between Two Line Segments:

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|.$$

The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$, parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

7. Projection of a line segment on a line

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then the projection of PQ on a line having direction cosines ℓ, m, n is

$$|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$$

8. Equation Of A Plane : General form: $ax + by + cz + d = 0$, where a, b, c are not all zero, $a, b, c, d \in R$.

(i) Normal form : $\ell x + my + nz = p$

(ii) Plane through the point (x_1, y_1, z_1) : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

(iii) Intercept Form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

(iv) Vector form: $(\vec{r} - \vec{a}).\vec{n} = 0$ or $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

(v) Any plane parallel to the given plane ax + by + cz + d = 0 is ax + by + cz + λ = 0.

Distance between ax + by + cz + d_1 = 0 and ax + by + cz + d_2 = 0 is = $\dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

(vi) **Equation of a plane passing through a given point & parallel to the given vectors:**

$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.

or $\vec{r}.(\vec{b}\times\vec{c}) = \vec{a}.(\vec{b}\times\vec{c})$ (non parametric form)

9. A Plane & A Point

(i) Distance of the point (x′, y′, z′) from the plane ax + by + cz+ d = 0 is given by $\dfrac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}$.

(ii) Length of the perpendicular from a point ($\vec{a}$) to plane $\vec{r}.\vec{n}$ = d is given by p = $\dfrac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}$.

(iii) Foot (x′, y′, z′) of perpendicular drawn from the point (x_1, y_1, z_1) to the plane ax + by + cz + d = 0 is given by $\dfrac{x'-x_1}{a} = \dfrac{y'-y_1}{b} = \dfrac{z'-z_1}{c} = -\dfrac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(iv) **To find image of a point w.r.t. a plane:**
Let P (x_1, y_1, z_1) is a given point and ax + by + cz + d = 0 is given plane Let (x′, y′, z′) is the image point. then $\dfrac{x'-x_1}{a} = \dfrac{y'-y_1}{b} = \dfrac{z'-z_1}{c} = -2\dfrac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

10. Angle Between Two Planes:

$$\cos\theta = \left|\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}\right|$$

Planes are perpendicular if aa′ + bb′ + cc′ = 0 and planes are parallel if $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

The angle θ between the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by, $\cos\theta = \dfrac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1|.|\vec{n}_2|}$

Planes are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$, λ is a scalar

11. Angle Bisectors

(i) The equations of the planes bisecting the angle between two given planes
$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(ii) Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0 \Rightarrow$ origin lies in acute angle

12. Family of Planes

(i) Any plane through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y + c_1z + d_1 + \lambda (a_2x + b_2y + c_2z + d_2) = 0$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}\cdot\vec{n}_1 = d_1$ & $\vec{r}\cdot\vec{n}_2 = d_2$ is $\vec{r}\cdot(n_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

13. **Area of triangle** : From two vector $\vec{AB}$ and $\vec{AC}$. Then area is given by $\frac{1}{2}|\vec{AB} \times \vec{AC}|$

14. Volume Of A Tetrahedron: Volume of a tetrahedron with vertices A (x_1, y_1, z_1), B(x_2, y_2, z_2), C (x_3, y_3, z_3) and D (x_4, y_4, z_4) is given by $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

A LINE

1. Equation Of A Line

(i) A straight line is intersection of two planes.
it is reprsented by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + +c_2z + d_2 = 0$.

(ii) Symmetric form : $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r$.

(iii) Vector equation: $\vec{r} = \vec{a} + \lambda\vec{b}$

(vi) Reduction of cartesion form of equation of a line to vector form & vice versa

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \Leftrightarrow \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

2. Angle Between A Plane And A Line:

(i) If θ is the angle between line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$, then $\sin\theta = \left|\frac{a\ell + bm + cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{\ell^2+m^2+n^2}}\right|$.

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

(iii) Condition for perpendicularity $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$, $\vec{b} \times \vec{n} = 0$

(iv) Condition for parallel $a\ell + bm + cn = 0$ $\vec{b}\cdot\vec{n} = 0$

3. Condition For A Line To Lie In A Plane

(i) Cartesian form: Line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie in a plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) Vector form: Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}.\vec{n} = d$ if $\vec{b}.\vec{n} = 0$ & $\vec{a}.\vec{n} = d$

4. Skew Lines:

(i) The straight lines which are not parallel and non–coplanar i.e. non–intersecting are called skew lines. If $\Delta = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, then lines are skew.

(iii) Vector Form: For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ to be skew $(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1) \neq 0$

(iv) Shortest distance between lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is $d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$.

5. Sphere

General equation of a sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. $(-u, -v, -w)$ is the centre and $\sqrt{u^2+v^2+w^2-d}$ is the radius of the sphere.

SOLUTION OF TRIANGLE

1. Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

2. Cosine Formula: (i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ (ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ (iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

3. Projection Formula: (i) $a = b\cos C + c\cos B$ (ii) $b = c\cos A + a\cos C$ (iii) $c = a\cos B + b\cos A$

4. Napier's Analogy - tangent rule:

(i) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ (ii) $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$ (iii) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

5. Trigonometric Functions of Half Angles:

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ where $s = \frac{a+b+c}{2}$ is semi perimetre of triangle.

(iv) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

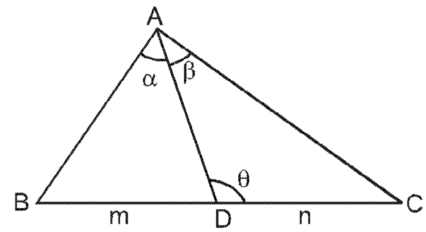
6. Area of Triangle (Δ) : $\Delta = \frac{1}{2}$ ab sin C = $\frac{1}{2}$ bc sin A = $\frac{1}{2}$ ca sin B = $\sqrt{s(s-a)(s-b)(s-c)}$

7. m - n Rule:

If BD : DC = m : n, then

$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

$= n\cot B - m\cot C$



8. Radius of Circumcirlce :

$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$

9. Radius of The Incircle :

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(iii) $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

(iv) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. Radius of The Ex-Circles :

(i) $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s\tan\frac{A}{2}$; $r_2 = s\tan\frac{B}{2}$; $r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

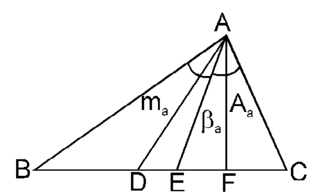
(iv) $r_1 = 4R\sin\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}$

11. Length of Angle Bisectors, Medians & Altitudes :

(i) Length of an angle bisector from the angle A = $\beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$;

(ii) Length of median from the angle A = $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

& (iii) Length of altitude from the angle A = $A_a = \frac{2\Delta}{a}$



12. The Distances of The Special Points from Vertices and Sides of Triangle:

(i) Circumcentre (O) : OA = R & O_a = R cos A **(ii)** Incentre (I) : IA = r cosec $\frac{A}{2}$ & I_a = r

(iii) Excentre (I_1) : $I_1 A = r_1$ cosec $\frac{A}{2}$ **(iv)** Orthocentre : HA = 2R cos A & H_a = 2R cos B cos C

(v) Centroid (G) : $GA = \frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ & $G_a = \frac{2\Delta}{3a}$

13. Orthocentre and Pedal Triangle:

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.

(i) Its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(ii) Its sides are $a \cos A = R \sin 2A$,
$b \cos B = R \sin 2B$ and
$c \cos C = R \sin 2C$

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

14. Excentral Triangle:

The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle.

(i) Δ ABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(ii) Its angles are $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ & $\frac{\pi}{2} - \frac{C}{2}$.

(iii) Its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ & $4R\cos\frac{C}{2}$.

(iv) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

(v) Incentre I of Δ ABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

15. Distance Between Special Points :

(i) Distance between circumcentre and orthocentre $OH^2 = R^2(1 - 8\cos A \cos B \cos C)$

(ii) Distance between circumcentre and incentre $OI^2 = R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = R^2 - 2Rr$

(iii) Distance between circumcentre and centroid $OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$

INVERSE TRIGONOMETRIC FUNCTIONS

1. Principal Values & Domains of Inverse Trigonometric/Circular Functions:

	Function		Domain	Range
(i)	$y = \sin^{-1}x$	where	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii)	$y = \cos^{-1}x$	where	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii)	$y = \tan^{-1}x$	where	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv)	$y = \text{cosec}^{-1}x$	where	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
(v)	$y = \sec^{-1}x$	where	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi$; $y \ne \frac{\pi}{2}$
(vi)	$y = \cot^{-1}x$	where	$x \in R$	$0 < y < \pi$

P - 2 (i) $\sin^{-1}(\sin x) = x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (ii) $\cos^{-1}(\cos x) = x$; $0 \le x \le \pi$

(iii) $\tan^{-1}(\tan x) = x; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$ (iv) $\cot^{-1}(\cot x) = x; \quad 0 < x < \pi$

(v) $\sec^{-1}(\sec x) = x; \quad 0 \le x \le \pi, \ x \ne \frac{\pi}{2}$ (vi) $\text{cosec}^{-1}(\text{cosec } x) = x; \quad x \ne 0, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

P - 3 (i) $\sin^{-1}(-x) = -\sin^{-1}x, \quad -1 \le x \le 1$ (ii) $\tan^{-1}(-x) = -\tan^{-1}x, \quad x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x, \quad -1 \le x \le 1$ (iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x, \quad x \in R$

P - 5 (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \ -1 \le x \le 1$ (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \ x \in R$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, \ |x| \ge 1$

2. Identities of Addition and Substraction:

I - 1 (i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right], \ x \ge 0, \ y \ge 0 \ \& \ (x^2 + y^2) \le 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right], \ x \ge 0, \ y \ge 0 \ \& \ x^2 + y^2 > 1$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\ \sqrt{1-y^2}\right], \ x \ge 0, \ y \ge 0$

(iii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}, \ x > 0, \ y > 0 \ \& \ xy < 1$

$= \pi + \tan^{-1}\frac{x+y}{1-xy}, \ x > 0, \ y > 0 \ \& \ xy > 1 \ = \frac{\pi}{2}, \ x > 0, \ y > 0 \ \& \ xy = 1$

I - 2 (i) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right], \ x \ge 0, \ y \ge 0$

(ii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\ \sqrt{1-y^2}\right], \ x \ge 0, \ y \ge 0, \ x \le y$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}, \ x \ge 0, \ y \ge 0$

I - 3 (i) $$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} 2\sin^{-1}x & \text{if } |x| \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } x > \frac{1}{\sqrt{2}} \\ -\left(\pi + 2\sin^{-1}x\right) & \text{if } x < -\frac{1}{\sqrt{2}} \end{cases}$$

(ii) $$\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & \text{if } 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \le x < 0 \end{cases}$$

(iii) $$\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -\left(\pi - 2\tan^{-1}x\right) & \text{if } x > 1 \end{cases}$$

(iv) $\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$

(v) $\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if, $x > 0$, $y > 0$, $z > 0$ & $(xy + yz + zx) < 1$

NOTE:

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ then $xy + yz + zx = 1$

(iii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(iv) $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$