# 6

# Application of Derivatives

## Short Answer Type Questions

**Q. 1** A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is propotional to the surface. Prove that the radius is decreasing at a constant rate.

**Thinking Process**

*First, let V be the volume of the ball and S be the surface area of the ball and then by using $\frac{dV}{dt} \propto S$, we can prove the required result.*

**Sol.** We have, rate of decrease of the volume of spherical ball of salt at any instant is $\propto$ surface.

Let the radius of the spherical ball of the salt be $r$.

$\therefore$ Volume of the ball $(V) = \frac{4}{3}\pi r^3$

and surface area $(S) = 4\pi r^2$

$\because \quad \frac{dV}{dt} \propto S \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \propto 4\pi r^2$

$\Rightarrow \quad \frac{4}{3}\pi \cdot 3r^2 . \frac{dr}{dt} \propto 4\pi r^2 \quad \Rightarrow \quad \frac{dr}{dt} \propto \frac{4\pi r^2}{4\pi r^2}$

$\Rightarrow \quad \frac{dr}{dt} = k \cdot 1$ [where, $k$ is the proportionality constant]

$\Rightarrow \quad \frac{dr}{dt} = k$

Hence, the radius of ball is decreasing at a constant rate.

**Q. 2** If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius.

**Sol.** Let the radius of circle $= r$ And area of the circle, $A = \pi r^2$

$\therefore \quad \frac{d}{dt}A = \frac{d}{dt}\pi r^2$

$\Rightarrow \quad \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$ ...(i)

Since, the area of a circle increases at a uniform rate, then

$$\frac{dA}{dt} = k \qquad \text{...(ii)}$$

where, $k$ is a constant.

From Eqs. (i) and (ii), $$2\pi r \cdot \frac{dr}{dt} = k$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{k}{2\pi r} = \frac{k}{2\pi} \cdot \left(\frac{1}{r}\right) \qquad \text{...(iii)}$$

Let the perimeter, $$P = 2\pi r$$

$$\therefore \quad \frac{dP}{dt} = \frac{d}{dt} \cdot 2\pi r \Rightarrow \frac{dP}{dt} = 2\pi \cdot \frac{dr}{dt}$$

$$= 2\pi \cdot \frac{k}{2\pi} \cdot \frac{1}{r} = \frac{k}{r} \qquad \text{[using Eq. (iii)]}$$

$$\Rightarrow \quad \frac{dP}{dt} \propto \frac{1}{r}$$

**Hence proved.**

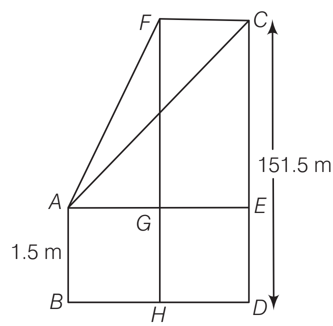
**Q. 3** A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, how fast is the string being let out, when the kite is 250 m away from the boy who is flying the kite, if the height of boy is 1.5 m?

**Sol.** We have, height $(h) = 151.5$ m, speed of kite $(v) = 10$ m/s

Let $CD$ be the height of kite and $AB$ be the height of boy.

Let $DB = x \text{ m} = EA$ and $AC = 250$ m

$$\therefore \quad \frac{dx}{dt} = 10 \text{ m/s}$$




From the figure, we see that

$$EC = 151.5 - 1.5 = 150 \text{ m}$$

and $$AE = x$$

Also, $$AC = 250 \text{ m}$$

In right angled $\Delta CEA$,

$$AE^2 + EC^2 = AC^2$$

$$\Rightarrow \quad x^2 + (150)^2 = y^2 \qquad \text{... (i)}$$

$$\Rightarrow \quad x^2 + (150)^2 = (250)^2$$

$$\Rightarrow \quad x^2 = (250)^2 - (150)^2$$

$$= (250 + 150)(250 - 150)$$

$$= 400 \times 100$$

$$\therefore \quad x = 20 \times 10 = 200$$

From Eq. (i), on differentiating w.r.t. $t$, we get

$$2x \cdot \frac{dx}{dt} + 0 = 2y \frac{dy}{dt}$$

$$\Rightarrow \quad 2y \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt}$$

$$= \frac{200}{250} \cdot 10 = 8 \text{ m/s} \qquad \left[\because \frac{dx}{dt} = 10 \text{ m/s}\right]$$

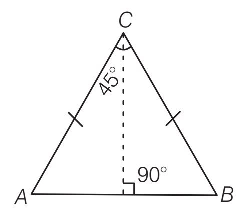
So, the required rate at which the string is being let out is 8 m/s.

**Q. 4** Two men $A$ and $B$ start with velocities $v$ at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, then find the rate at which they are being separated.

**Thinking Process**

*By drawing figure such that men start moving at a point C, A and B are separating points, then draw perpendicular from that point C to AB to get D. Now, get the value of $\angle ACD$ in terms of $x$ and y, then by using $\frac{dy}{dt}$ get desired result. [ let $AC = BC = x$ and $AB = y$]*

**Sol.** Let two men start from the point $C$ with velocity $v$ each at the same time.




Also, $\angle BCA = 45°$

Since, $A$ and $B$ are moving with same velocity $v$, so they will cover same distance in same time.

Therefore, $\Delta ABC$ is an isosceles triangle with $AC = BC$.

Now, draw $CD \perp AB$.

Let at any instant $t$, the distance between them is $AB$.

Let $AC = BC = x$ and $AB = y$

In $\Delta ACD$ and $\Delta DCB$,

$$\angle CAD = \angle CBD \qquad [\because AC = BC]$$

$$\angle CDA = \angle CDB = 90°$$

$$\therefore \quad \angle ACD = \angle DCB$$

$$\text{or} \quad \angle ACD = \frac{1}{2} \times \angle ACB$$

$$\Rightarrow \quad \angle ACD = \frac{1}{2} \times 45°$$

$$\Rightarrow \quad \angle ACD = \frac{\pi}{8}$$

$$\therefore \quad \sin\frac{\pi}{8} = \frac{AD}{AC}$$

$$\Rightarrow \quad \sin\frac{\pi}{8} = \frac{y/2}{x} \qquad [\because AD = y/2]$$

$$\Rightarrow \quad \frac{y}{2} = x \sin\frac{\pi}{8}$$

$$\Rightarrow \quad y = 2x \cdot \sin\frac{\pi}{8}$$

Now, differentiating both sides w.r.t. $t$, we get

$$\frac{dy}{dt} = 2 \cdot \sin\frac{\pi}{8} \cdot \frac{dx}{dt}$$

$$= 2 \cdot \sin\frac{\pi}{8} . v \qquad \left[\because v = \frac{dx}{dt}\right]$$

$$= 2v \cdot \frac{\sqrt{2 - \sqrt{2}}}{2} \qquad \left[\because \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}\right]$$

$$= \sqrt{2 - \sqrt{2}}\, v \text{ unit/s}$$

which is the rate at which $A$ and $B$ are being separated.

**Q. 5** **Find an angle $\theta$, where $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.**

**Sol.** Let $\theta$ increases twice as fast as its sine.

$$\Rightarrow \quad \theta = 2 \sin \theta$$

Now, on differentiating both sides w.r.t. $t$, we get

$$\frac{d\theta}{dt} = 2 \cdot \cos\theta \cdot \frac{d\theta}{dt} \Rightarrow 1 = 2\cos\theta$$

$$\Rightarrow \quad \frac{1}{2} = \cos\theta \Rightarrow \cos\theta = \cos\frac{\pi}{3}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

So, the required angle is $\frac{\pi}{3}$.

**Q. 6** **Find the approximate value of $(1.999)^5$.**

**Sol.** Let $x = 2$

and $\Delta x = -0.001$ $\quad [\because 2 - 0.001 = 1.999]$

Let $y = x^5$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = 5x^4$$

Now, $\Delta y = \frac{dy}{dx} \cdot \Delta x = 5x^4 \times \Delta x$

$$= 5 \times 2^4 \times [-0.001]$$

$$= -80 \times 0.001 = -0.080$$

$$\therefore \quad (1.999)^5 = y + \Delta y$$

$$= 2^5 + (-0.080)$$

$$= 32 - 0.080 = 31.920$$

**Q. 7** **Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm, respectively.**

**Sol.** Let internal radius $= r$ and external radius $= R$

$\therefore$ Volume of hollow spherical shell, $V = \frac{4}{3}\pi(R^3 - r^3)$

$$\Rightarrow \quad V = \frac{4}{3}\pi[(3.0005)^3 - (3)^3] \quad \text{... (i)}$$

Now, we shall use differentiation to get approximate value of $(3.0005)^3$.

Let $(3.0005)^3 = y + \Delta y$

and $x = 3, \Delta x = 0.0005$

Also, let $y = x^3$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = 3x^2$$

$$\therefore \quad \Delta y = \frac{dy}{dx} \times \Delta x = 3x^2 \times 0.0005$$

$$= 3 \times 3^2 \times 0.0005$$

$$= 27 \times 0.0005 = 0.0135$$

Also, $(3.0005)^3 = y + \Delta y$

$$= 3^3 + 0.0135 = 27.0135$$

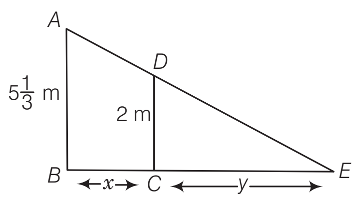
$\therefore$ $$V = \frac{4}{3}\pi\,[27.0135 - 27.000] \quad \text{[using Eq. (i)]}$$

$$= \frac{4}{3}\pi\,[0.0135] = 4\pi \times (0.0045)$$

$$= 0.0180\pi \text{ cm}^3$$

**Q. 8** **A man, 2 m tall, walks at the rate of $1\frac{2}{3}$ m/s towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the tip of his shadow moving and at what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light?**

**Sol.** Let $AB$ be the street light post and $CD$ be the height of man *i.e.*, $CD = 2$ m.




Let $BC = x$ m, $CE = y$ m and $\frac{dx}{dt} = \frac{-5}{3}$ m/s

From $\Delta ABE$ and $\Delta DCE$, we see that

$$\Delta ABE \sim \Delta DCE \quad \text{[by AAA similarity]}$$

$\therefore$ $$\frac{AB}{DC} = \frac{BE}{CE} \Rightarrow \frac{\frac{16}{3}}{2} = \frac{x+y}{y}$$

$\Rightarrow$ $$\frac{16}{6} = \frac{x+y}{y}$$

$\Rightarrow$ $$16y = 6x + 6y \Rightarrow 10y = 6x$$

$\Rightarrow$ $$y = \frac{3}{5}x$$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dy}{dt} = \frac{3}{5}\cdot\frac{dx}{dt} = \frac{3}{5}\cdot\left(-1\frac{2}{3}\right)$$

[since, man is moving towards the light post]

$$= \frac{3}{5}\cdot\left(\frac{-5}{3}\right) = -1 \text{ m/s}$$

Let $z = x + y$

Now, differentiating both sides w.r.t. $t$, we get

$$\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -\left(\frac{5}{3} + 1\right)$$

$$= -\frac{8}{3} = -2\frac{2}{3} \text{ m/s}$$

Hence, the tip of shadow is moving at the rate of $2\frac{2}{3}$ m/s towards the light source and length of the shadow is decreasing at the rate of 1 m/s.

**Q. 9** **A swimming pool is to be drained for cleaning. If $L$ represents the number of litres of water in the pool $t$ seconds after the pool has been plugged off to drain and $L = 200\,(10 - t)^2$. How fast is the water running out at the end of 5 s and what is the average rate at which the water flows out during the first 5 s?**

**Sol.** Let $L$ represents the number of litres of water in the pool $t$ seconds after the pool has been plugged off to drain, then

$$L = 200\,(10 - t)^2$$

$\therefore$ Rate at which the water is running out $= -\dfrac{dL}{dt}$

$$\frac{dL}{dt} = -200\cdot 2\,(10 - t)\cdot(-1)$$
$$= 400\,(10 - t)$$

Rate at which the water is running out at the end of 5 s

$$= 400\,(10 - 5)$$
$$= 2000\text{ L/s} = \text{Final rate}$$

Since, initial rate $= -\left(\dfrac{dL}{dt}\right)_{t=0} = 4000$ L/s

$\therefore$ Average rate during 5 s $= \dfrac{\text{Initial rate} + \text{Final rate}}{2}$

$$= \frac{4000 + 2000}{2}$$
$$= 3000\text{ L/s}$$

**Q. 10** **The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.**

**Sol.** Let the side of a cube be $x$ unit.

$\therefore$ Volume of cube $(V) = x^3$

On differentiating both side w.r.t. $t$, we get

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} = k \qquad \text{[constant]}$$

$$\Rightarrow \frac{dx}{dt} = \frac{k}{3x^2} \qquad \text{... (i)}$$

Also, surface area of cube, $S = 6x^2$

On differentiating w.r.t. $t$, we get

$$\frac{dS}{dt} = 12x\cdot\frac{dx}{dt}$$

$$\Rightarrow \frac{dS}{dt} = 12\,x\cdot\frac{k}{3x^2} \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \frac{dS}{dt} = \frac{12k}{3x} = 4\left(\frac{k}{x}\right)$$

$$\Rightarrow \frac{dS}{dt} \propto \frac{1}{x}$$

Hence, the surface area of the cube varies inversely as the length of the side.

**Q. 11** If $x$ and $y$ are the sides of two squares such that $y = x - x^2$, then find the rate of change of the area of second square with respect to the area of first square.

**Thinking Process**

*First, let $A_1$ and $A_2$ be the areas of two squares and get their values in one variable and then by using $dA_1/dt$ and $dA_2/dt$ get the value of $dA_2/dA_1$*

**Sol.** Since, $x$ and $y$ are the sides of two squares such that $y = x - x^2$.

$\therefore$ Area of the first square $(A_1) = x^2$

and area of the second square $(A_2) = y^2 = (x - x^2)^2$

$$\therefore \quad \frac{dA_2}{dt} = \frac{d}{dt}(x - x^2)^2 = 2(x - x^2)\left(\frac{dx}{dt} - 2x \cdot \frac{dx}{dt}\right)$$

$$= \frac{dx}{dt}(1 - 2x)\,2(x - x^2)$$

and

$$\frac{dA_1}{dt} = \frac{d}{dt}x^2 = 2x \cdot \frac{dx}{dt}$$

$$\therefore \quad \frac{dA_2}{dA_1} = \frac{dA_2/dt}{dA_1/dt} = \frac{\frac{dx}{dt} \cdot (1 - 2x)(2x - 2x^2)}{2x \cdot \frac{dx}{dt}}$$

$$= \frac{(1 - 2x)\,2x(1 - x)}{2x}$$

$$= (1 - 2x)(1 - x)$$

$$= 1 - x - 2x + 2x^2$$

$$= 2x^2 - 3x + 1$$

**Q. 12** Find the condition that curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

**Thinking Process**

*First, get the intersection point of the curve and then get the slopes of both the curves at that point. Then, by using $m_1 \cdot m_2 = -1$, get the required condition.*

**Sol.** Given, equation of curves are $2x = y^2$ ... (i)

and $2xy = k$ ... (ii)

$\Rightarrow \quad y = \dfrac{k}{2x}$ [from Eq. (ii)]

From Eq. (i), $\quad 2x = \left(\dfrac{k}{2x}\right)^2$

$\Rightarrow \quad 8x^3 = k^2$

$\Rightarrow \quad x^3 = \dfrac{1}{8}k^2$

$\Rightarrow \quad x = \dfrac{1}{2}k^{2/3}$

$$\therefore \quad y = \frac{k}{2x} = \frac{k}{2 \cdot \frac{1}{2}k^{2/3}} = k^{1/3}$$

Thus, we get point of intersection of curves which is $\left(\dfrac{1}{2}k^{2/3}, k^{1/3}\right)$.

From Eqs. (i) and (ii),

$$2 = 2y\frac{dy}{dx}$$

and $$2\left[x \cdot \frac{dy}{dx} + y \cdot 1\right] = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{1}{y}$$

and $$\left(\frac{dy}{dx}\right) = \frac{-2y}{2x} = -\frac{y}{x}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3},\, k^{1/3}\right)} = \frac{1}{k^{1/3}} \quad \text{[say } m_1\text{]}$$

and $$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3},\, k^{1/3}\right)} = \frac{-k^{1/3}}{\frac{1}{2}k^{2/3}} = -2k^{-1/3} \quad \text{[say } m_2\text{]}$$

Since, the curves intersect orthogonally.

*i.e.,* $$m_1 \cdot m_2 = -1$$

$\Rightarrow$ $$\frac{1}{k^{1/3}} \cdot (-2k^{-1/3}) = -1$$

$\Rightarrow$ $$-2\,k^{-2/3} = -1$$

$\Rightarrow$ $$\frac{2}{k^{2/3}} = 1$$

$\Rightarrow$ $$k^{2/3} = 2$$

$\therefore$ $$k^2 = 8$$

which is the required condition.

## Q. 13 Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

### Thinking Process

*First, find the intersection points of curves and then equate the slopes of both the curves at the obtained point.*

**Sol.** Given equation of curves are

$$xy = 4 \quad \text{...(i)}$$

and $$x^2 + y^2 = 8 \quad \text{...(ii)}$$

$\Rightarrow$ $$x \cdot \frac{dy}{dx} + y = 0$$

and $$2x + 2y\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{-y}{x}$$

and $$\frac{dy}{dx} = \frac{-2x}{2y}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{-y}{x} = m_1 \quad \text{[say]}$$

and $$\frac{dy}{dx} = \frac{-x}{y} = m_2 \quad \text{[say]}$$

Since, both the curves should have same slope.

$\therefore$ $$\frac{-y}{x} = \frac{-x}{y} \Rightarrow -y^2 = -x^2$$

$\Rightarrow$ $$x^2 = y^2 \quad \text{...(iii)}$$

Using the value of $x^2$ in Eq. (ii), we get

$$y^2 + y^2 = 8$$

$$\Rightarrow \quad y^2 = 4 \Rightarrow y = \pm 2$$

For $y = 2$, $x = \frac{4}{2} = 2$

and for $y = -2$, $x = \frac{4}{-2} = -2$

Thus, the required points of intersection are (2, 2) and (– 2, – 2).

For (2, 2), $\quad m_1 = \frac{-y}{x} = \frac{-2}{2} = -1$

and $\quad m_2 = \frac{-x}{y} = \frac{-2}{2} = -1$

$\because \quad m_1 = m_2$

For (–2,–2), $\quad m_1 = \frac{-y}{x} = \frac{-(-2)}{-2} = -1$

and $\quad m_2 = \frac{-x}{y} = \frac{-(-2)}{-2} = -1$

Thus, for both the intersection points, we see that slope of both the curves are same. Hence, the curves touch each other.

**Q. 14** **Find the coordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.**

**Sol.** We have, $\quad \sqrt{x} + \sqrt{y} = 4 \quad$ ...(i)

$$\Rightarrow \quad x^{1/2} + y^{1/2} = 4$$

$$\Rightarrow \quad \frac{1}{2} \cdot \frac{1}{x^{1/2}} + \frac{1}{2} \cdot \frac{1}{y^{1/2}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2} \cdot x^{-1/2} \quad 2 \cdot y^{1/2}$$

$$= -\sqrt{\frac{y}{x}}$$

Since, tangent is equally inclined to the axes.

$$\therefore \quad \frac{dy}{dx} = \pm 1$$

$$\Rightarrow \quad -\sqrt{\frac{y}{x}} = \pm 1$$

$$\Rightarrow \quad \frac{y}{x} = 1 \Rightarrow y = x$$

From Eq. (i), $\quad \sqrt{y} + \sqrt{y} = 4$

$$\Rightarrow \quad 2\sqrt{y} = 4$$

$$\Rightarrow \quad 4y = 16$$

$$\therefore \quad y = 4 \text{ and } x = 4$$

When y=4, then $x = 4$

So, the required coordinates are (4, 4).

**Q. 15** Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

**Sol.** We have, $y = 4 - x^2$ ...(i)

and $y = x^2$ ... (ii)

$\Rightarrow$ $\frac{dy}{dx} = -2x$

and $\frac{dy}{dx} = 2x$

$\Rightarrow$ $m_1 = -2x$

and $m_2 = 2x$

From Eqs. (i) and (ii), $x^2 = 4 - x^2$

$\Rightarrow$ $2x^2 = 4$

$\Rightarrow$ $x^2 = 2$

$\Rightarrow$ $x = \pm\sqrt{2}$

$\therefore$ $y = x^2 = (\pm\sqrt{2})^2 = 2$

So, the points of intersection are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.

For point $(+\sqrt{2}, 2)$, $m_1 = -2x = -2\cdot\sqrt{2} = -2\sqrt{2}$

and $m_2 = 2x = 2\sqrt{2}$

and for point $(\sqrt{2}, 2)$, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-2\sqrt{2} - 2\sqrt{2}}{1 - 2\sqrt{2}\cdot 2\sqrt{2}}\right| = \left|\frac{-4\sqrt{2}}{-7}\right|$

$\therefore$ $\theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

**Q. 16** Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

**Sol.** We have, $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$

Since, both the curves touch each other at (1, 2) *i.e.,* curves are passing through (1, 2).

$\therefore$ $2y\cdot\frac{dy}{dx} = 4$

and $2x + 2y\frac{dy}{dx} = 6$

$\Rightarrow$ $\frac{dy}{dx} = \frac{4}{2y}$

and $\frac{dy}{dx} = \frac{6 - 2x}{2y}$

$\Rightarrow$ $\left(\frac{dy}{dx}\right)_{(1,2)} = \frac{4}{4} = 1$

and $\left(\frac{dy}{dx}\right)_{(1,2)} = \frac{6 - 2\cdot 1}{2\cdot 2} = \frac{4}{4} = 1$

$\Rightarrow$ $m_1 = 1$ and $m_2 = 1$

Thus, we see that slope of both the curves are equal to each other *i.e.,* $m_1 = m_2 = 1$ at the point (1, 2).

Hence, both the curves touch each other.

**Q. 17** **Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.**

**Sol.** Given equation of the curve is

$$3x^2 - y^2 = 8 \qquad \text{...(i)}$$

On differentiating both sides w.r.t. $x$, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y}$$

$$\Rightarrow \quad m_1 = \frac{3x}{y} \qquad \text{[say]}$$

and $\qquad$ slope of normal $(m_2) = \frac{-1}{m_1} = \frac{-y}{3x}$ $\qquad$ ...(ii)

Since, slope of normal to the curve should be equal to the slope of line $x + 3y = 4$, which is parallel to curve.

For line, $\qquad y = \frac{4 - x}{3} = \frac{-x}{3} + \frac{4}{3}$

$\Rightarrow$ $\qquad$ Slope of the line $(m_3) = \frac{-1}{3}$

$$\therefore \quad m_2 = m_3$$

$$\Rightarrow \quad \frac{-y}{3x} = -\frac{1}{3}$$

$$\Rightarrow \quad -3y = -3x$$

$$\Rightarrow \quad y = x \qquad \text{...(iii)}$$

On substituting the value of $y$ in Eq. (i), we get

$$3x^2 - x^2 = 8$$

$$\Rightarrow \quad x^2 = 4$$

$$\Rightarrow \quad x = \pm 2$$

For $x = 2$, $\qquad y = 2$ $\qquad$ [using Eq. (iii)]

and for $x = -2$, $\qquad y = -2$ $\qquad$ [using Eq. (iii)]

Thus,the points at which normal to the curve are parallel to the line $x + 3y = 4$ are (2, 2) and (– 2, – 2).

Required equations of normal are

$$y - 2 = m_2(x - 2) \quad \text{and} \quad y + 2 = m_2(x + 2)$$

$$\Rightarrow \quad y - 2 = \frac{-2}{6}(x - 2) \quad \text{and} \quad y + 2 = \frac{-2}{6}(x + 2)$$

$$\Rightarrow \quad 3y - 6 = -x + 2 \quad \text{and} \quad 3y + 6 = -x - 2$$

$$\Rightarrow \quad 3y + x = +8 \quad \text{and} \quad 3y + x = -8$$

So, the required equations are $3y + x = \pm 8$.

**Q. 18** **At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the $Y$-axis?**

**Sol.** Given, equation of curve which is

$$x^2 + y^2 - 2x - 4y + 1 = 0 \qquad \text{... (i)}$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(2y-4) = 2-2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(1-x)}{2(y-2)}$$

Since, the tangents are parallel to the $Y$-axis *i.e.,* $\tan\theta = \tan 90° = \frac{dy}{dx}$.

$$\therefore \quad \frac{1-x}{y-2} = \frac{1}{0}$$

$$\Rightarrow \quad y-2 = 0$$

$$\Rightarrow \quad y = 2$$

For $y = 2$ from Eq. (i), we get

$$x^2 + 2^2 - 2x - 4 \times 2 + 1 = 0$$

$$\Rightarrow \quad x^2 - 2x - 3 = 0$$

$$\Rightarrow \quad x^2 - 3x + x - 3 = 0$$

$$\Rightarrow \quad x(x-3) + 1(x-3) = 0$$

$$\Rightarrow \quad (x+1)(x-3) = 0$$

$$\therefore \quad x = -1,\ x = 3$$

So, the required points are $(-1, 2)$ and $(3, 2)$.

**Q. 19 Show that the line $\frac{x}{a} + \frac{y}{b} = 1$, touches the curve $y = b \cdot e^{-x/a}$ at the point, where the curve intersects the axis of $Y$.**

**Sol.** We have the equation of line given by $\frac{x}{a} + \frac{y}{b} = 1$, which touches the curve $y = b \cdot e^{-x/a}$ at the point, where the curve intersects the axis of $Y$ *i.e.,* $x = 0$.

$$\therefore \quad y = b \cdot e^{-0/a} = b \qquad [\because e^0 = 1]$$

So, the point of intersection of the curve with $Y$-axis is $(0, b)$.

Now, slope of the given line at $(0, b)$ is given by

$$\frac{1}{a} \cdot 1 + \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{a} \cdot b$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{a} \cdot b = \frac{-b}{a} = m_1 \qquad \text{[say]}$$

Also, the slope of the curve at $(0, b)$ is

$$\frac{dy}{dx} = b \cdot e^{-x/a} \cdot \frac{-1}{a}$$

$$\frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$$

$$\left(\frac{dy}{dx}\right)_{(0,\, b)} = \frac{-b}{a} e^{-0} = \frac{-b}{a} = m_2 \qquad \text{[say]}$$

Since, $\quad m_1 = m_2 = \frac{-b}{a}$

Hence, the line touches the curve at the point, where the curve intersects the axis of $Y$.

**Q. 20** Show that $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1 + x^2} - x)$ is increasing in $R$.

**Thinking Process**

*If $f'(x) \geq 0$, then we can say that $f(x)$ is increasing function. Use this condition to show the desired result.*

**Sol.** We have,
$$f(x) = 2x + \cot^{-1} x + \log(\sqrt{1 + x^2} - x)$$
$$\therefore \quad f'(x) = 2 + \left(\frac{-1}{1 + x^2}\right) + \frac{1}{(\sqrt{1 + x^2} - x)}\left(\frac{1}{2\sqrt{1 + x^2}} \cdot 2x - 1\right)$$
$$= 2 - \frac{1}{1 + x^2} + \frac{1}{(\sqrt{1 + x^2} - x)} \cdot \frac{(x - \sqrt{1 + x^2})}{\sqrt{1 + x^2}}$$
$$= 2 - \frac{1}{1 + x^2} - \frac{1}{\sqrt{1 + x^2}}$$
$$= \frac{2 + 2x^2 - 1 - \sqrt{1 + x^2}}{1 + x^2} = \frac{1 + 2x^2 - \sqrt{1 + x^2}}{1 + x^2}$$

For increasing function, $f'(x) \geq 0$
$$\Rightarrow \quad \frac{1 + 2x^2 - \sqrt{1 + x^2}}{1 + x^2} \geq 0$$
$$\Rightarrow \quad 1 + 2x^2 \geq \sqrt{1 + x^2}$$
$$\Rightarrow \quad (1 + 2x^2)^2 \geq 1 + x^2$$
$$\Rightarrow \quad 1 + 4x^4 + 4x^2 \geq 1 + x^2$$
$$\Rightarrow \quad 4x^4 + 3x^2 \geq 0$$
$$\Rightarrow \quad x^2(4x^2 + 3) \geq 0$$

which is true for any real value of $x$.

Hence, $f(x)$ is increasing in $R$.

**Q. 21** Show that for $a \geq 1$, $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ is decreasing in $R$.

**Thinking Process**

*If $f'(x) \leq 0$, then we can say that $f(x)$ is a decreasing function. So, use this condition to show the result.*

**Sol.** We have, $a \geq 1$,
$$f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$$
$$\therefore \quad f'(x) = \sqrt{3}\cos x - (-\sin x) - 2a$$
$$= \sqrt{3}\cos x + \sin x - 2a$$
$$= 2\left[\frac{\sqrt{3}}{2} \cdot \cos x + \frac{1}{2} \cdot \sin x\right] - 2a$$
$$= 2\left[\cos\frac{\pi}{6} \cdot \cos x + \sin\frac{\pi}{6} \cdot \sin x\right] - 2a$$
$$= 2\left(\cos\frac{\pi}{6} - x\right) - 2a$$
$$[\because \cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B]$$
$$= 2\left[\left(\cos\frac{\pi}{6} - x\right) - a\right]$$

We know that, $\cos x \in [-1, 1]$

and $a \geq 1$

So, $2\left[\cos\left(\frac{\pi}{6} - x\right) - a\right] \leq 0$

$\therefore$ $f'(x) \leq 0$

Hence, $f(x)$ is a decreasing function in $R$.

**Q. 22** Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

**Sol.** We have, $f(x) = \tan^{-1}(\sin x + \cos x)$

$$\therefore \quad f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot (\cos x - \sin x)$$

$$= \frac{1}{1 + \sin^2 x + \cos^2 x + 2\sin x \cdot \cos x}(\cos x - \sin x)$$

$$= \frac{1}{(2 + \sin 2x)}(\cos x - \sin x)$$

$[\because \sin 2x = 2\sin x \cos x \text{ and } \sin^2 x + \cos^2 x = 1]$

For $f'(x) \geq 0$,

$$\frac{1}{(2 + \sin 2x)} \cdot (\cos x - \sin x) \geq 0$$

$\Rightarrow$ $\cos x - \sin x \geq 0$ $\left[\because (2 + \sin 2x) \geq 0 \text{ in } \left(0, \frac{\pi}{4}\right)\right]$

$\Rightarrow$ $\cos x \geq \sin x$

which is true, if $x \in \left(0, \frac{\pi}{4}\right)$.

Hence, $f(x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

**Q. 23** At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum? Also, find the maximum slope.

**Sol.** We have, $y = -x^3 + 3x^2 + 9x - 27$

$\therefore$ $\frac{dy}{dx} = -3x^2 + 6x + 9 =$ Slope of tangent to the curve

Now, $\frac{d^2y}{dx^2} = -6x + 6$

For $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$,

$$-6x + 6 = 0$$

$$\Rightarrow \quad x = \frac{-6}{-6} = 1$$

$$\therefore \quad \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = -6 < 0$$

So, the slope of tangent to the curve is maximum, when $x = 1$.

For $x = 1$, $\left(\frac{dy}{dx}\right)_{(x=1)} = -3 \cdot 1^2 + 6 \cdot 1 + 9 = 12$,

which is maximum slope.

Also, for $x = 1, \; y = -1^3 + 3\cdot 1^2 + 9\cdot 1 - 27$

$= -1 + 3 + 9 - 27$

$= -16$

So, the required point is (1, – 16).

**Q. 24** Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \dfrac{\pi}{6}$.

**Sol.** We have, $f(x) = \sin x + \sqrt{3}\cos x$

$\therefore$ $f'(x) = \cos x + \sqrt{3}(-\sin x)$

$= \cos x - \sqrt{3}\sin x$

For $f'(x) = 0$, $\cos x = \sqrt{3}\sin x$

$\Rightarrow$ $\tan x = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6}$

$\Rightarrow$ $x = \dfrac{\pi}{6}$

Again, differentiating $f'(x)$, we get

$f''(x) = -\sin x - \sqrt{3}\cos x$

At $x = \dfrac{\pi}{6}$, $f''(x) = -\sin\dfrac{\pi}{6} - \sqrt{3}\cos\dfrac{\pi}{6}$

$= -\dfrac{1}{2} - \sqrt{3}\cdot\dfrac{\sqrt{3}}{2}$

$= -\dfrac{1}{2} - \dfrac{3}{2} = -2 < 0$

Hence, at $x = \dfrac{\pi}{6}$, $f(x)$ has maximum value at $\dfrac{\pi}{6}$ is the point of local maxima.

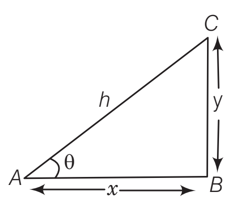
## Long Answer Type Questions

**Q. 25** If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, then show that the area of triangle is maximum, when the angle between them is $\dfrac{\pi}{3}$.

**Sol.** Let $ABC$ be a triangle with $AC = h$, $AB = x$ and $BC = y$.

Also, $\angle CAB = \theta$

Let $h + x = k$ ...(i)




$\therefore$ $\cos\theta = \dfrac{x}{h}$

$\Rightarrow$ $x = h\cos\theta$

$\Rightarrow$ $h + h\cos\theta = k$ [using Eq. (i)]

$$\Rightarrow \quad h(1+\cos\theta) = k$$

$$\Rightarrow \quad h = \frac{k}{(1+\cos\theta)} \quad \ldots(ii)$$

Also, area of $\Delta ABC = \frac{1}{2}(AB \cdot BC)$

$$A = \frac{1}{2} \cdot x \cdot y$$

$$= \frac{1}{2} h\cos\theta \cdot h\sin\theta \quad \left[\because \sin\theta = \frac{y}{h}\right]$$

$$= \frac{1}{2} h^2 \sin\theta \cdot \cos\theta$$

$$= \frac{2h^2}{4} \sin\theta \cdot \cos\theta$$

$$= \frac{1}{4} h^2 \sin 2\theta \quad \ldots(iii)$$

Since, $h = \frac{k}{1+\cos\theta}$

$$\therefore \quad A = \frac{1}{4}\left(\frac{k}{1+\cos\theta}\right)^2 \cdot \sin 2\theta$$

$$\Rightarrow \quad A = \frac{k^2}{4} \cdot \frac{\sin 2\theta}{(1+\cos\theta)^2} \quad \ldots(iv)$$

$$\therefore \quad \frac{dA}{d\theta} = \frac{k^2}{4}\left[\frac{(1+\cos\theta)^2 \cdot \cos 2\theta \cdot 2 - \sin 2\theta \cdot 2(1+\cos\theta) \cdot (0-\sin\theta)}{(1+\cos\theta)^4}\right]$$

$$= \frac{k^2}{4}\left\{\frac{2(1+\cos\theta)[(1+\cos\theta) \cdot \cos 2\theta + \sin 2\theta\,(\sin\theta)}{(1+\cos\theta)^4}\right\}$$

$$= \frac{k^2}{4} \cdot \frac{2}{(1+\cos\theta)^3}[(1+\cos\theta) \cdot \cos 2\theta + 2\sin^2\theta \cdot \cos\theta]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}[(1+\cos\theta)(1-2\sin^2\theta) + 2\sin^2\theta \cdot \cos\theta]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}[1+\cos\theta - 2\sin^2\theta - 2\sin^2\theta \cdot \cos\theta + 2\sin^2\theta \cdot \cos\theta]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}[(1+\cos\theta) - 2\sin^2\theta]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}[1+\cos\theta - 2 + 2\cos^2\theta]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta + \cos\theta - 1) \quad \ldots(v)$$

For $\frac{dA}{d\theta} = 0$,

$$\frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta + \cos\theta - 1) = 0$$

$$\Rightarrow \quad 2\cos^2\theta + \cos\theta - 1 = 0$$

$$\Rightarrow \quad 2\cos^2\theta + 2\cos\theta - \cos\theta - 1 = 0$$

$$\Rightarrow \quad 2\cos\theta(\cos\theta + 1) - 1(\cos\theta + 1) = 0$$

$$\Rightarrow \quad (2\cos\theta - 1)(\cos\theta + 1) = 0$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1$$

$$\Rightarrow \quad \theta = \frac{\pi}{3} \quad \text{[possible]}$$

$$\text{or} \quad \theta = 2n\pi \pm \pi \quad \text{[not possible]}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

Again, differentiating w.r.t. $\theta$ in Eq. (v), we get

$$\frac{d}{d\theta}\left(\frac{dA}{d\theta}\right) = \frac{d}{d\theta}\left[\frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta + \cos\theta - 1)\right]$$

$$\therefore \quad \frac{d^2A}{d\theta^2} = \frac{d}{d\theta}\left[\frac{k^2(2\cos\theta - 1)(1+\cos\theta)}{2(1+\cos\theta)^3}\right] = \frac{d}{d\theta}\left[\frac{k^2}{2}\cdot\frac{(2\cos\theta-1)}{(1+\cos\theta)^2}\right]$$

$$= \frac{k^2}{2}\left[\frac{(1+\cos\theta)^2\cdot(-2\sin\theta) - 2(1+\cos\theta)\cdot(-\sin\theta)(2\cos\theta - 1)}{(1+\cos\theta)^4}\right]$$

$$= \frac{k^2}{2}\left[\frac{(1+\cos\theta)\cdot[1+\cos\theta](-2\sin\theta) + 2\sin\theta\,(2\cos\theta - 1)}{(1+\cos\theta)^4}\right]$$

$$= \frac{k^2}{2}\left[\frac{-2\sin\theta - 2\sin\theta\cdot\cos\theta + 4\sin\theta\cdot\cos\theta - 2\sin\theta}{(1+\cos\theta)^3}\right]$$

$$= \frac{k^2}{2}\left[\frac{-4\sin\theta - \sin 2\theta + 2\sin 2\theta}{(1+\cos\theta)^3}\right] = \frac{k^2}{2}\left[\frac{\sin 2\theta - 4\sin\theta}{(1+\cos\theta)^3}\right]$$

$$\therefore \quad \left(\frac{d^2A}{d\theta^2}\right)_{\text{at } \theta = \frac{\pi}{3}} = \frac{k^2}{2}\left[\frac{\sin\frac{2\pi}{3} - 4\sin\frac{\pi}{3}}{\left(1+\cos\frac{\pi}{3}\right)^3}\right] = \frac{k^2}{2}\left[\frac{\frac{\sqrt{3}}{2} - \frac{4\sqrt{3}}{2}}{\left(1+\frac{1}{2}\right)^3}\right]$$

$$= \frac{k^2}{2}\left[\frac{-3\sqrt{3}\cdot 8}{2\cdot 27}\right] = -k^2\left(\frac{2\sqrt{3}}{9}\right)$$

which is less than zero.

Hence, area of the right angled triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

**Q. 26 Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also, find the corresponding local maximum and local minimum values.**

**Sol.** Given that, $\quad f(x) = x^5 - 5x^4 + 5x^3 - 1$

On differentiating w.r.t. $x$, we get

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

For maxima or minima, $\quad f'(x) = 0$

$$\Rightarrow \quad 5x^4 - 20x^3 + 15x^2 = 0$$

$$\Rightarrow \quad 5x^2(x^2 - 4x + 3) = 0$$

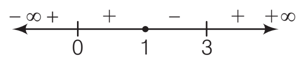
$$\Rightarrow \quad 5x^2(x^2 - 3x - x + 3) = 0$$

$$\Rightarrow \quad 5x^2[x(x-3) - 1(x-3)] = 0$$

$$\Rightarrow \quad 5x^2[(x-1)(x-3)] = 0$$

$$\therefore \quad x = 0, 1, 3$$

Sign scheme for $\frac{dy}{dx} = 5x^2(x-1)(x-3)$




So, $y$ has maximum value at $x = 1$ and minimum value at $x = 3$.
At $x = 0$, $y$ has neither maximum nor minimum value.

$\therefore$ Maximum value of $y = 1 - 5 + 5 - 1 = 0$

and minimum value $= (3)^5 - 5(3)^4 + 5(3)^3 - 1$

$= 243 - 81 \times 5 - 27 \times 5 - 1 = -298$

**Q. 27** **A telephone company in a town has 500 subscribers on its list and collects fixed charges of ₹ 300 per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of ₹ 1 per one subscriber will discontinue the service. Find what increase will bring maximum profit?**

**Sol.** Consider that company increases the annual subscription by ₹ $x$.
So, $x$ subscribes will discontinue the service.
$\therefore$ Total revenue of company after the increment is given by

$$R(x) = (500 - x)(300 + x)$$
$$= 15 \times 10^4 + 500x - 300x - x^2$$
$$= -x^2 + 200x + 150000$$

On differentiating both sides w.r.t. $x$, we get

$$R'(x) = -2x + 200$$

Now, $R'(x) = 0$

$\Rightarrow$ $2x = 200 \Rightarrow x = 100$

$\therefore$ $R''(x) = -2 < 0$

So, $R(x)$ is maximum when $x = 100$.
Hence, the company should increase the subscription fee by ₹ 100, so that it has maximum profit.

**Q. 28** **If the straight line $x \cos\alpha + y \sin\alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2 \cos^2\alpha + b^2 \sin^2\alpha = p^2$.**

**Sol.** Given, line is $x \cos\alpha + y \sin\alpha = p$ ... (i)

and curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow$ $b^2x^2 + a^2y^2 = a^2b^2$ ...(ii)

Now, differentiating Eq. (ii) w.r.t. $x$, we get

$$b^2 \cdot 2x + a^2 \cdot 2y \cdot \frac{dy}{dx} = 0$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{-2b^2x}{2a^2y} = \frac{-xb^2}{ya^2}$ ... (iii)

From Eq. (i), $y \sin\alpha = p - x \cos\alpha$

$\Rightarrow$ $y = -x \cot\alpha + \frac{p}{\sin\alpha}$

Thus, slope of the line is $(-\cot\alpha)$.

So, the given equation of line will be tangent to the Eq. (ii), if $\left(-\frac{x}{y}\cdot\frac{b^2}{a^2}\right) = (-\cot\alpha)$

$$\Rightarrow \quad \frac{x}{a^2\cos\alpha} = \frac{y}{b^2\sin\alpha} = k \quad \text{[say]}$$

$$\Rightarrow \quad x = ka^2\cos\alpha$$

and $$y = b^2 k\sin\alpha$$

So, the line $x\cos\alpha + y\sin\alpha = p$ will touch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ at point $(ka^2\cos\alpha, kb^2\sin\alpha)$.

From Eq. (i), $$ka^2\cos^2\alpha + kb^2\sin^2\alpha = p$$

$$\Rightarrow \quad a^2\cos^2\alpha + b^2\sin^2\alpha = \frac{p}{k}$$

$$\Rightarrow \quad (a^2\cos^2\alpha + b^2\sin^2\alpha)^2 = \frac{p^2}{k^2} \quad \text{...(iv)}$$

From Eq. (ii), $$b^2k^2a^4\cos^2\alpha + a^2k^2b^4\sin^2\alpha = a^2b^2$$

$$\Rightarrow \quad k^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = 1$$

$$\Rightarrow \quad (a^2\cos^2\alpha + b^2\sin^2\alpha) = \frac{1}{k^2} \quad \text{... (v)}$$

On dividing Eq. (iv) by Eq. (v), we get

$$a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$$ **Hence proved.**

**Alternate Method**

We know that, if a line $y = mx + c$ touches ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

the required condition is $c^2 = a^2m^2 + b^2$

Here, given equation of the line is

$$x\cos\alpha + y\sin\alpha = p$$

$$\Rightarrow \quad y = \frac{p - x\cos\alpha}{\sin\alpha}$$

$$= -x\cot\alpha + \frac{p}{\sin\alpha}$$

$$\Rightarrow \quad c = \frac{p}{\sin\alpha}$$

and $$m = -\cot\alpha$$

$$\therefore \quad \left(\frac{p}{\sin\alpha}\right)^2 = a^2(-\cot\alpha)^2 + b^2$$

$$\Rightarrow \quad \frac{p^2}{\sin^2\alpha} = a^2\frac{\cos^2\alpha}{\sin^2\alpha} + b^2$$

$$\Rightarrow \quad p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$ **Hence proved.**

**Q. 29** If an open box with square base is to be made of a given quantity of card board of area $c^2$, then show that the maximum volume of the box is $\dfrac{c^3}{6\sqrt{3}}$ cu units.

**Thinking Process**

*First, let the sides of box in $x$ and $y$ then find $\dfrac{dV}{dx}$ in terms c and $x$. Also, for $\dfrac{dV}{dx}=0$ get the value of $x$ and if $\dfrac{d^2V}{dx^2} < 0$ at the value of $x$, then by putting that value of $x$ in the equation of V, get the desired result.*

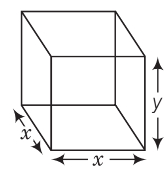
**Sol.** Let the length of side of the square base of open box be $x$ units and its height be $y$ units.

$\therefore$ Area of the metal used $= x^2 + 4xy$

$$\Rightarrow \quad x^2 + 4xy = c^2 \quad \text{[given]}$$

$$\Rightarrow \quad y = \frac{c^2 - x^2}{4x} \quad \text{...(i)}$$

Now, volume of the box $(V) = x^2 y$




$$\Rightarrow \quad V = x^2 \cdot \left(\frac{c^2 - x^2}{4x}\right)$$

$$= \frac{1}{4} x (c^2 - x^2)$$

$$= \frac{1}{4} (c^2 x - x^3)$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2) \quad \text{... (ii)}$$

Now,
$$\frac{dV}{dx} = 0 \Rightarrow c^2 = 3x^2$$

$$\Rightarrow \quad x^2 = \frac{c^2}{3}$$

$$\Rightarrow \quad x = \frac{c}{\sqrt{3}} \quad \text{[using positive sign]}$$

Again, differentiating Eq. (ii) w.r.t. $x$, we get

$$\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2} x < 0$$

$$\therefore \quad \left(\frac{d^2v}{dx^2}\right)_{\text{at } x = \frac{c}{\sqrt{3}}} = -\frac{3}{2} \cdot \left(\frac{c}{\sqrt{3}}\right) < 0$$

Thus, we see that volume $(V)$ is maximum at $x = \dfrac{c}{\sqrt{3}}$.

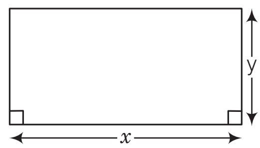
$\therefore$ Maximum volume of the box, $(V)_{x = \frac{c}{\sqrt{3}}} = \dfrac{1}{4}\left(c^2 . \dfrac{c}{\sqrt{3}} - \dfrac{c^3}{3\sqrt{3}}\right)$

$$= \frac{1}{4} \cdot \frac{(3c^3 - c^3)}{3\sqrt{3}} = \frac{1}{4} \cdot \frac{2c^3}{3\sqrt{3}}$$

$$= \frac{c^3}{6\sqrt{3}} \text{ cu units}$$

**Q. 30** **Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also, find the maximum volume.**

**Sol.** Let breadth and length of the rectangle be $x$ and $y$, respectively.




$\because$ Perimeter of the rectangle = 36 cm

$\Rightarrow \quad 2x + 2y = 36$

$\Rightarrow \quad x + y = 18$

$\Rightarrow \quad y = 18 - x \quad$ ... (i)

Let the rectangle is being revolved about its length $y$.

Then, volume ($V$) of resultant cylinder $= \pi x^2 \cdot y$

$\Rightarrow \quad V = \pi x^2 \cdot (18 - x) \quad [\because V = \pi r^2 h]$ [using Eq. (i)]

$= 18\pi x^2 - \pi x^3 = \pi [18x^2 - x^3]$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dV}{dx} = \pi (36x - 3x^2)$$

Now, $\quad \frac{dV}{dx} = 0$

$\Rightarrow \quad 36x = 3x^2$

$\Rightarrow \quad 3x^2 - 36x = 0$

$\Rightarrow \quad 3(x^2 - 12x) = 0$

$\Rightarrow \quad 3x(x - 12) = 0$

$\Rightarrow \quad x = 0, x = 12$

$\therefore \quad x = 12 \quad [\because, x \neq 0]$

Again, differentiating w.r.t. $x$, we get

$$\frac{d^2V}{dx^2} = \pi (36 - 6x)$$

$$\Rightarrow \quad \left(\frac{d^2V}{dx^2}\right)_{x = 12} = \pi(36 - 6 \times 12) = -36\pi < 0$$

At $x = 12$, volume of the resultant cylinder is the maximum.

So, the dimensions of rectangle are 12 cm and 6 cm, respectively. [using Eq. (i)]

$\therefore$ Maximum volume of resultant cylinder,

$$(V)_{x = 12} = \pi [18 \cdot (12)^2 - (12)^3]$$

$$= \pi [12^2 (18 - 12)]$$

$$= \pi \times 144 \times 6$$

$$= 864 \pi \text{ cm}^3$$

**Q. 31** I the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

**Sol.** Let length of one edge of cube be $x$ units and radius of sphere be $r$ units.

$\therefore$ Surface area of cube $= 6x^2$

and surface area of sphere $= 4\pi r^2$

Also, $6x^2 + 4\pi r^2 = k$ [constant, given]

$$\Rightarrow \quad 6x^2 = k - 4\pi r^2$$

$$\Rightarrow \quad x^2 = \frac{k - 4\pi r^2}{6}$$

$$\Rightarrow \quad x = \left[\frac{k - 4\pi r^2}{6}\right]^{1/2} \quad \text{... (i)}$$

Now, volume of cube $= x^3$

and volume of sphere $= \frac{4}{3}\pi r^3$

Let sum of volume of the cube and volume of the sphere be given by

$$S = x^3 + \frac{4}{3}\pi r^3 = \left[\frac{k - 4\pi r^2}{6}\right]^{3/2} + \frac{4}{3}\pi r^3$$

On differentiating both sides w.r.t. $r$, we get

$$\frac{dS}{dr} = \frac{3}{2}\left[\frac{k - 4\pi r^2}{6}\right]^{1/2} \cdot \left(\frac{-8\pi r}{6}\right) + \frac{12}{3}\pi r^2$$

$$= -2\pi r\left[\frac{k - 4\pi r^2}{6}\right]^{1/2} + 4\pi r^2 \quad \text{... (ii)}$$

$$= -2\pi r\left[\left\{\frac{k - 4\pi r^2}{6}\right\}^{1/2} - 2r\right]$$

Now, $\frac{dS}{dr} = 0$

$$\Rightarrow \quad r = 0 \text{ or } 2r = \left(\frac{k - 4\pi r^2}{6}\right)^{1/2}$$

$$\Rightarrow \quad 4r^2 = \frac{k - 4\pi r^2}{6} \Rightarrow 24r^2 = k - 4\pi r^2$$

$$\Rightarrow \quad 24r^2 + 4\pi r^2 = k \Rightarrow r^2\,[24 + 4\pi] = k$$

$$\therefore \quad r = 0 \text{ or } r = \sqrt{\frac{k}{24 + 4\pi}} = \frac{1}{2}\sqrt{\frac{k}{6 + \pi}}$$

We know that, $r \neq 0$

$$\therefore \quad r = \frac{1}{2}\sqrt{\frac{k}{6 + \pi}}$$

Again, differentiating w.r.t. $r$ in Eq. (ii), we get

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[-2\pi r\left\{\left(\frac{k - 4\pi r^2}{6}\right)^{1/2} + 4\pi r^2\right\}\right]$$

$$= -2\pi\left[r\cdot\frac{1}{2}\left(\frac{k-4\pi r^2}{6}\right)^{-1/2}\cdot\left(\frac{-8\pi r}{6}\right)+\left(\frac{k-4\pi r^2}{6}\right)^{1/2}\cdot 1\right]+4\pi\cdot 2r$$

$$= -2\pi\left[r\cdot\frac{1}{2\sqrt{\frac{k-4\pi r^2}{6}}}\cdot\left(\frac{-8\pi r}{6}\right)+\sqrt{\frac{k-4\pi r^2}{6}}\right]+8\pi r$$

$$= -2\pi\left[\frac{-8\pi r^2+12\left(k-\frac{4\pi r^2}{6}\right)}{12\sqrt{\frac{k-4\pi r^2}{6}}}\right]+8\pi r$$

$$= -2\pi\left[\frac{-48\pi r^2+72k-48\pi r^2}{72\sqrt{\frac{k-4\pi r^2}{6}}}\right]+8\pi r = -2\pi\left[\frac{-96\pi r^2+72k}{72\sqrt{\frac{k-4\pi r^2}{6}}}\right]+8\pi r>0$$

For $r=\frac{1}{2}\sqrt{\frac{k}{6+\pi}}$, then the sum of their volume is minimum.

For $r=\frac{1}{2}\sqrt{\frac{k}{6+\pi}}$,

$$x=\left[\frac{k-4\pi\cdot\frac{1}{4}\frac{k}{(6+\pi)}}{6}\right]^{1/2}$$

$$=\left[\frac{(6+\pi)k-\pi k}{6(6+\pi)}\right]^{1/2}=\left[\frac{k}{6+\pi}\right]^{1/2}=2r$$

Since, the sum of their volume is minimum when $x=2r$.

Hence, the ratio of an edge of cube to the diameter of the sphere is 1:1.

**Q. 32 If $AB$ is a diameter of a circle and $C$ is any point on the circle, then show that the area of $\Delta ABC$ is maximum, when it is isosceles.**

**Sol.** We have, $AB = 2r$

and $\angle ACB = 90°$ [since, angle in the semi-circle is always 90°]

Let $AC = x$ and $BC = y$

$\therefore$ $(2r)^2 = x^2 + y^2$

$\Rightarrow$ $y^2 = 4r^2 - x^2$

$\Rightarrow$ $y = \sqrt{4r^2 - x^2}$ ... (i)

Now, area of $\Delta ABC$, $A = \frac{1}{2}\times x\times y$

$= \frac{1}{2}\times x\times(4r^2-x^2)^{1/2}$ [using Eq. (i)]

Now, differentiating both sides w.r.t. $x$, we get

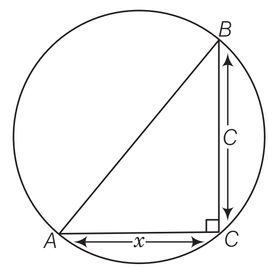
$$\frac{dA}{dx}=\frac{1}{2}\left[x\cdot\frac{1}{2}(4r^2-x^2)^{-1/2}\cdot(0-2x)+(4r^2-x^2)^{1/2}\cdot 1\right]$$

$$=\frac{1}{2}\left[\frac{-2x^2}{2\sqrt{4r^2-x^2}}+(4r^2-x^2)^{1/2}\right]$$

$$= \frac{1}{2}\left[\frac{-x^2}{\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2}\right]$$

$$= \frac{1}{2}\left[\frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}}\right] = \frac{1}{2}\left[\frac{-2x^2 + 4r^2}{\sqrt{4r^2 - x^2}}\right]$$

$$\Rightarrow \quad \frac{dA}{dx} = \left[\frac{(-x^2 + 2r^2)}{\sqrt{4r^2 - x^2}}\right]$$




Now, $$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad -x^2 + 2r^2 = 0$$

$$\Rightarrow \quad r^2 = \frac{1}{2}x^2$$

$$\Rightarrow \quad r = \frac{1}{\sqrt{2}}x$$

$$\therefore \quad x = r\sqrt{2}$$

Again, differentiating both sides w.r.t. $x$, we get

$$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}\cdot(-2x) + (2r^2 - x^2)\cdot\frac{1}{2}(4r^2 - x^2)^{-1/2}(-2x)}{(\sqrt{4r^2 - x^2})^2}$$

$$= \frac{-2x\left[\sqrt{4r^2 - x^2} + (2r^2 - x^2)\cdot\frac{1}{2\sqrt{4r^2 - x^2}}\right]}{(\sqrt{4r^2 - x^2})^2}$$

$$= \frac{-4x\cdot\left(\sqrt{4r^2 - x^2}\right)^2 + (2r^2 - x^2)(-2x)}{2\cdot(4r^2 - x^2)^{3/2}}$$

$$= \frac{-4x(4r^2 - x^2) + (2r^2 - x^2)\cdot(-2x)}{2\cdot(4r^2 - x^2)^{3/2}}$$

$$= \frac{-16xr^2 + 4x^3 + (2r^2 - x^2)(-2x)}{2\cdot(4r^2 - x^2)^{3/2}}$$

$$\left(\frac{d^2A}{dx^2}\right)_{x = r\sqrt{2}} = \frac{-16\cdot r\sqrt{2}\cdot r^2 + 4\cdot(r\sqrt{2})^3 + [2r^2 - (r\sqrt{2})^2]\cdot(-2\cdot r\sqrt{2})}{2\cdot(4r^2 - 2r^2)^{3/2}} \qquad [\because x = r\sqrt{2}]$$

$$= \frac{-16\sqrt{2}\cdot r^3 + 8\sqrt{2}r^3}{2(2r^2)^{3/2}} = \frac{8\sqrt{2}\, r^2\,[r - 2r]}{4r^3}$$

$$= \frac{-8\sqrt{2}\, r^3}{4r^3} = -2\sqrt{2} < 0$$

For $x = r\sqrt{2}$, the area of triangle is maximum.

For $x = r\sqrt{2}$, $$y = \sqrt{4r^2 - (r\sqrt{2})^2} = \sqrt{2r^2} = r\sqrt{2}$$

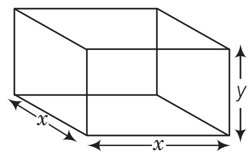
Since, $$x = r\sqrt{2} = y$$

Hence, the triangle is isosceles.

**Q. 33** A metal box with a square base and vertical sides is to contain $1024 \text{ cm}^3$. If the material for the top and bottom costs ₹ 5per $\text{cm}^2$ and the material for the sides costs ₹ 2.50 per $\text{cm}^2$. Then, find the least cost of the box.

**Sol.** Since, volume of the box = $1024 \text{ cm}^3$

Let length of the side of square base be $x$ cm and height of the box be $y$ cm.




$\therefore$ Volume of the box $(V) = x^2 \cdot y = 1024$

Since, $x^2 y = 1024 \Rightarrow y = \dfrac{1024}{x^2}$

Let $C$ denotes the cost of the box.

$$\therefore \quad C = 2x^2 \times 5 + 4xy \times 2.50$$
$$= 10x^2 + 10xy = 10x(x + y)$$
$$= 10x\left(x + \frac{1024}{x^2}\right)$$
$$= \frac{10x}{x^2}(x^3 + 1024)$$
$$\Rightarrow \quad C = 10x^2 + \frac{10240}{x} \quad \text{... (i)}$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dC}{dx} = 20x + 10240(-x)^{-2}$$
$$= 20x - \frac{10240}{x^2} \quad \text{...(ii)}$$

Now, $\dfrac{dC}{dx} = 0$

$$\Rightarrow \quad 20x = \frac{10240}{x^2}$$
$$\Rightarrow \quad 20x^3 = 10240$$
$$\Rightarrow \quad x^3 = 512 = 8^3 \Rightarrow x = 8$$

Again, differentiating Eq. (ii) w.r.t. $x$, we get

$$\frac{d^2C}{dx^2} = 20 - 10240(-2) \cdot \frac{1}{x^3}$$
$$= 20 + \frac{20480}{x^3} > 0$$
$$\therefore \quad \left(\frac{d^2C}{dx^2}\right)_{x=8} = 20 + \frac{20480}{512} = 60 > 0$$

For $x = 8$, cost is minimum and the corresponding least cost of the box,

$$C(8) = 10 \cdot 8^2 + \frac{10240}{8}$$
$$= 640 + 1280 = 1920$$

$\therefore$ Least cost = ₹ 1920

**Q. 34** The sum of surface areas of a rectangular parallelopiped with sides $x$, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if $x$ is equal to three times the radius of the sphere. Also, find the minimum value of the sum of their volumes.

**Sol.** We have given that, the sum of the surface areas of a rectangular parallelopiped with sides $x$, $2x$ and $\frac{x}{3}$ and a sphere is constant.

Let $S$ be the sum of both the surface area.

$$\therefore \quad S = 2\left(x \cdot 2x + 2x \cdot \frac{x}{3} + \frac{x}{3} \cdot x\right) + 4\pi r^2 = k$$

$$k = 2\left[2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right] + 4\pi r^2$$

$$= 2\,[3x^2] + 4\pi r^2 = 6x^2 + 4\pi r^2$$

$$\Rightarrow \quad 4\pi r^2 = k - 6x^2$$

$$\Rightarrow \quad r^2 = \frac{k - 6x^2}{4\pi}$$

$$\Rightarrow \quad r = \sqrt{\frac{k - 6x^2}{4\pi}} \qquad \text{... (i)}$$

Let $V$ denotes the volume of both the parallelopiped and the sphere.

Then, $$V = 2x \cdot x \cdot \frac{x}{3} + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\left(\frac{k - 6x^2}{4\pi}\right)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \frac{1}{8\pi^{3/2}}(k - 6x^2)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{1}{6\sqrt{\pi}}(k - 6x^2)^{3/2} \qquad \text{...(ii)}$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dV}{dx} = \frac{2}{3} \cdot 3x^2 + \frac{1}{6\sqrt{\pi}} \cdot \frac{3}{2}(k - 6x^2)^{1/2} \cdot (-12x)$$

$$= 2x^2 - \frac{12x}{4\sqrt{\pi}}\sqrt{k - 6x^2}$$

$$= 2x^2 - \frac{3x}{\sqrt{\pi}}(k - 6x^2)^{1/2} \qquad \text{...(iii)}$$

$$\because \quad \frac{dV}{dx} = 0$$

$$\Rightarrow \quad 2x^2 = \frac{3x}{\sqrt{\pi}}(k - 6x^2)^{1/2}$$

$$\Rightarrow \quad 4x^4 = \frac{9x^2}{\pi}(k - 6x^2)$$

$$\Rightarrow \quad 4\pi x^4 = 9kx^2 - 54x^4$$

$$\Rightarrow \quad 4\pi x^4 + 54x^4 = 9kx^2$$

$$\Rightarrow \quad x^4[4\pi + 54] = 9 \cdot k \cdot x^2$$

$$\Rightarrow \quad x^2 = \frac{9k}{4\pi + 54}$$

$$\Rightarrow \quad x = 3 \cdot \sqrt{\frac{k}{4\pi + 54}} \qquad \text{...(iv)}$$

Again, differentiating Eq. (iii) w.r.t. $x$, we get

$$\frac{d^2V}{dx^2} = 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1}{2}(k - 6x^2)^{-1/2} \cdot (-12x) + (k - 6x^2)^{1/2} \cdot 1\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}[-6x^2 \cdot (k - 6x^2)^{-1/2} + (k - 6x^2)^{1/2}]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + k - 6x^2}{\sqrt{k - 6x^2}}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{k - 12x^2}{\sqrt{k - 6x^2}}\right]$$

Now, $$\left(\frac{d^2V}{dx^2}\right)_{x = 3\cdot\sqrt{\frac{k}{4\pi + 54}}} = 4 \cdot 3\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{k - 12 \cdot 9 \cdot \frac{k}{4\pi + 54}}{\sqrt{k - \frac{6 \cdot 9 \cdot k}{4\pi + 54}}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{k - \frac{108k}{4\pi + 54}}{\sqrt{k - \frac{54k}{4\pi + 54}}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{4k\pi + 54k - 108k / 4\pi + 54}{\sqrt{4k\pi + 54k - 54k / 4\pi + 54}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{4k\pi - 54k}{\sqrt{4k\pi}\sqrt{4\pi + 54}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{6}{\sqrt{\pi}}\left[\frac{k(2\pi - 27)}{\sqrt{k}\sqrt{16\pi^2 + 216\pi}}\right]$$

$$\left[\text{since, } (2\pi - 27) < 0 \Rightarrow \frac{d^2V}{dx^2} > 0; k > 0\right]$$

For $x = 3\sqrt{\frac{k}{4\pi + 54}}$, the sum of volumes is minimum.

For $x = 3\sqrt{\frac{k}{4\pi + 54}}$, then $r = \sqrt{\frac{k - 6x^2}{4\pi}}$ [using Eq. (i)]

$$= \frac{1}{2\sqrt{\pi}}\sqrt{k - 6 \cdot \frac{9k}{4\pi + 54}}$$

$$= \frac{1}{2\sqrt{\pi}} \cdot \sqrt{\frac{4k\pi + 54k - 54k}{4\pi + 54}}$$

$$= \frac{1}{2\sqrt{\pi}}\sqrt{\frac{4k\pi}{4\pi + 54}} = \frac{\sqrt{k}}{\sqrt{4\pi + 54}} = \frac{1}{3}x$$

$\Rightarrow$ $x = 3r$ **Hence proved.**

$\therefore$ Minimum sum of volume,

$$V_{\left(x = 3\cdot\sqrt{\frac{k}{4\pi + 54}}\right)} = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \left(\frac{1}{3}x\right)^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \frac{x^3}{27} = \frac{2}{3}x^3\left(1 + \frac{2\pi}{27}\right)$$

## Objective Type Questions

**Q. 35** If the sides of an equilateral triangle are increasing at the rate of 2 cm/s then the rate at which the area increases, when side is 10 cm, is

(a) $10\text{ cm}^2/\text{s}$ (b) $\sqrt{3}\text{ cm}^2/\text{s}$

(c) $10\sqrt{3}\text{ cm}^2/\text{s}$ (d) $\frac{10}{3}\text{ cm}^2/\text{s}$

**Sol.** *(c)* Let the side of an equilateral triangle be $x$ cm.

$\therefore$ Area of equilateral triangle, $A = \frac{\sqrt{3}}{4}x^2$ ...(i)

Also, $\frac{dx}{dt} = 2\text{cm/s}$

On differentiating Eq. (i) w.r.t. $t$, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \cdot 2x \cdot \frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{4} \cdot 2 \cdot 10 \cdot 2 \qquad \left[\because x = 10 \text{ and } \frac{dx}{dt} = 2\right]$$

$$= 10\sqrt{3}\text{ cm}^2/\text{s}$$

**Q. 36** A ladder, 5 m long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/s, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 m from the wall is

(a) $\frac{1}{10}$ rad/s (b) $\frac{1}{20}$ rad/s

(c) 20 rad/s (d) 10 rad/s

**Sol.** *(b)* Let the angle between floor and the ladder be $\theta$.

Let $AB = x$ cm and $BC = y$ cm

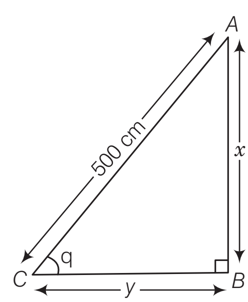
$\therefore$ $\sin\theta = \frac{x}{500}$ and $\cos\theta = \frac{y}{500}$

$\Rightarrow$ $x = 500\sin\theta$ and $y = 500\cos\theta$

Also, $\frac{dx}{dt} = 10$ cm/s

$\Rightarrow$ $500 \cdot \cos\theta \cdot \frac{d\theta}{dt} = 10$ cm/s

$\Rightarrow$ $\frac{d\theta}{dt} = \frac{10}{500\cos\theta} = \frac{1}{50\cos\theta}$




For $y = 2\text{ m} = 200\text{ cm}$,

$$\frac{d\theta}{dt} = \frac{1}{50 \cdot \frac{y}{500}} = \frac{10}{y}$$

$$= \frac{10}{200} = \frac{1}{20}\text{ rad/s}$$

**Q. 37** The curve $y = x^{1/5}$ has at (0, 0)

(a) a vertical tangent (parallel to $Y$-axis)

(b) a horizontal tangent (parallel to $X$-axis)

(c) an oblique tangent

(d) no tangent

**Sol.** *(a)* We have,

$$y = x^{1/5}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5}x^{-4/5}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(0,0)} = \frac{1}{5} \times (0)^{-4/5} = \infty$$

So, the curve $y = x^{1/5}$ has a vertical tangent at (0, 0), which is parallel to $Y$-axis.

**Q. 38** The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(a) $3x - y = 8$

(b) $3x + y + 8 = 0$

(c) $x + 3y \pm 8 = 0$

(d) $x + 3y = 0$

**Sol.** *(c)* We have, the equation of the curve is $3x^2 - y^2 = 8$ ...(i)

Also, the given equation of the line is $x + 3y = 8$.

$$\Rightarrow \quad 3y = 8 - x$$

$$\Rightarrow \quad y = -\frac{x}{3} + \frac{8}{3}$$

Thus, slope of the line is $-\frac{1}{3}$ which should be equal to slope of the equation of normal to the curve.

On differentiating Eq. (i) w.r.t. $x$, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y} = \text{Slope of the curve}$$

Now, slope of normal to the curve $= -\dfrac{1}{\left(\dfrac{dy}{dx}\right)}$

$$= -\frac{1}{\left(\dfrac{3x}{y}\right)} = -\frac{y}{3x}$$

$$\therefore \quad -\left(\frac{y}{3x}\right) = -\frac{1}{3}$$

$$\Rightarrow \quad -3y = -3x$$

$$\Rightarrow \quad y = x$$

On substituting the value of the given equation of the curve, we get

$$3x^2 - x^2 = 8$$

$$\Rightarrow \quad x^2 = \frac{8}{2}$$

$$\Rightarrow \quad x = \pm 2$$

For $x = 2$, $\quad 3(2)^2 - y^2 = 8$

$\Rightarrow \quad y^2 = 4$

$\Rightarrow \quad y = \pm 2$

and for $x = -2$, $\quad 3(-2)^2 - y^2 = 8$

$\Rightarrow \quad y = \pm 2$

So, the points at which normals are parallel to the given line are $(\pm 2, \pm 2)$.

Hence, the equation of normal at $(\pm 2, \pm 2)$ is

$$y - (\pm 2) = -\frac{1}{3}[x - (\pm 2)]$$

$\Rightarrow \quad 3[y - (\pm 2)] = -[x - (\pm 2)]$

$\Rightarrow \quad x + 3y \pm 8 = 0$

**Q. 39** If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of $a$ is

(a) 1 (b) 0 (c) $-6$ (d) 6

**Sol.** *(d)* We have, $\quad ay + x^2 = 7$ and $x^3 = y$

On differentiating w.r.t. $x$ in both equations, we get

$$a \cdot \frac{dy}{dx} + 2x = 0 \quad \text{and } 3x^2 = \frac{dy}{dx}$$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a}$ and $\frac{dy}{dx} = 3x^2$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-2}{a} = m_1$

and $\quad \left(\frac{dy}{dx}\right)_{(1,1)} = 3 \cdot 1 = 3 = m_2$

Since, the curves cut orthogonally at (1, 1).

$\therefore \quad m_1 \cdot m_2 = -1$

$\Rightarrow \quad \left(\frac{-2}{a}\right) \cdot 3 = -1$

$\therefore \quad a = 6$

**Q. 40** If $y = x^4 - 10$ and $x$ changes from 2 to 1.99, then what is the change in $y$?

(a) 0.32 (b) 0.032 (c) 5.68 (d) 5.968

**Sol.** *(a)* We have, $y = x^4 - 10 \Rightarrow \frac{dy}{dx} = 4x^3$

and $\quad \Delta x = 2.00 - 1.99 = 0.01$

$\therefore \quad \Delta y = \frac{dy}{dx} \times \Delta x$

$= 4x^3 \times \Delta x$

$= 4 \times 2^3 \times 0.01$

$= 32 \times 0.01 = 0.32$

So, the approximate change in $y$ is 0.32.

**Q. 41** The equation of tangent to the curve $y(1+x^2)=2-x$, where it crosses $X$-axis, is

(a) $x+5y=2$ (b) $x-5y=2$
(c) $5x-y=2$ (d) $5x+y=2$

**Sol.** *(a)* We have, equation of the curve $y(1+x^2)=2-x$ ...(i)

$\therefore \quad y\cdot(0+2x)+(1+x^2)\cdot\dfrac{dy}{dx}=0-1$ [on differentiating w.r.t. $x$]

$\Rightarrow \quad 2xy+(1+x^2)\dfrac{dy}{dx}=-1$

$\Rightarrow \quad \dfrac{dy}{dx}=\dfrac{-1-2xy}{1+x^2}$ ...(ii)

Since, the given curve passes through $X$-axis *i.e.*, $y=0$.

$\therefore \quad 0(1+x^2)=2-x$ [using Eq. (i)]

$\Rightarrow \quad x=2$

So, the curve passes through the point (2, 0).

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(2,\,0)}=\dfrac{-1-2\times0}{1+2^2}=-\dfrac{1}{5}=$ Slope of the curve

$\therefore$ Slope of tangent to the curve $=-\dfrac{1}{5}$

$\therefore$ Equation of tangent of the curve passing through (2, 0) is

$$y-0=-\frac{1}{5}(x-2)$$

$\Rightarrow \quad y+\dfrac{x}{5}=+\dfrac{2}{5}$

$\Rightarrow \quad 5y+x=2$

**Q. 42** The points at which the tangents to the curve $y=x^3-12x+18$ are parallel to $X$-axis are

(a) $(2,-2),(-2,-34)$ (b) $(2,34),(-2,0)$
(c) $(0,34),(-2,0)$ (d) $(2,2),(-2,34)$

**Sol.** *(d)* The given equation of curve is

$$y=x^3-12x+18$$

$\therefore \quad \dfrac{dy}{dx}=3x^2-12$ [on differentiating w.r.t. $x$]

So, the slope of line parallel to the $X$-axis.

$\therefore \quad \left(\dfrac{dy}{dx}\right)=0$

$\Rightarrow \quad 3x^2-12=0$

$\Rightarrow \quad x^2=\dfrac{12}{3}=4$

$\therefore \quad x=\pm 2$

For $x=2$, $\quad y=2^3-12\times2+18=2$

and for $x=-2$, $y=(-2)^3-12(-2)+18=34$

So, the points are $(2,2)$ and $(-2,34)$.

**Q. 43** The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets $X$-axis at

(a) (0, 1) (b) $\left(-\frac{1}{2}, 0\right)$ (c) (2, 0) (d) (0, 2)

**Sol.** *(b)* The equation of curve is $y = e^{2x}$

Since, it passes through the point (0, 1).

$$\therefore \quad \frac{dy}{dx} = e^{2x} \cdot 2 = 2 \cdot e^{2x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,\,1)} = 2 \cdot e^{2 \cdot 0} = 2 = \text{Slope of tangent to the curve}$$

$\therefore$ Equation of tangent is $y - 1 = 2(x - 0)$

$$\Rightarrow \quad y = 2x + 1$$

Since, tangent to curve $y = e^{2x}$ at the point (0, 1) meets $X$-axis *i.e.,* $y = 0$.

$$\therefore \quad 0 = 2x + 1 \Rightarrow x = -\frac{1}{2}$$

So, the required point is $\left(\frac{-1}{2}, 0\right)$.

**Q. 44** The slope of tangent to the curve $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$ at the point (2, – 1) is

(a) $\frac{22}{7}$ (b) $\frac{6}{7}$ (c) $-\frac{6}{7}$ (d) – 6

**Sol.** *(b)* Equation of curve is given by

$$x = t^2 + 3t - 8 \quad \text{and} \quad y = 2t^2 - 2t - 5.$$

$$\therefore \quad \frac{dx}{dt} = 2t + 3 \quad \text{and} \quad \frac{dy}{dt} = 4t - 2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t - 2}{2t + 3} \qquad \text{...(i)}$$

Since, the curve passes through the point (2, – 1).

$$\therefore \quad 2 = t^2 + 3t - 8$$

$$\text{and} \quad -1 = 2t^2 - 2t - 5$$

$$\Rightarrow \quad t^2 + 3t - 10 = 0$$

$$\text{and} \quad 2t^2 - 2t - 4 = 0$$

$$\Rightarrow \quad t^2 + 5t - 2t - 10 = 0$$

$$\text{and} \quad 2t^2 + 2t - 4t - 4 = 0$$

$$\Rightarrow \quad t(t + 5) - 2(t + 5) = 0$$

$$\text{and} \quad 2t(t + 1) - 4(t + 1) = 0$$

$$\Rightarrow \quad (t - 2)(t + 5) = 0$$

$$\text{and} \quad (2t - 4)(t + 1) = 0$$

$$\Rightarrow \quad t = 2, -5 \text{ and } t = -1, 2$$

$$\Rightarrow \quad t = 2$$

$\therefore$ Slope of tangent,

$$\left(\frac{dy}{dx}\right)_{\text{at } t\,=\,2} = \frac{4 \times 2 - 2}{2 \times 2 + 3} = \frac{6}{7} \qquad \text{[using Eq. (i)]}$$

**Q. 45** Two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect at an angle of

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

**Sol.** ***(c)*** Equation of two curves are given by

$$x^3 - 3xy^2 + 2 = 0$$

and $$3x^2y - y^3 - 2 = 0 \quad \text{[on differentiating w.r.t. } x\text{]}$$

$\Rightarrow$ $$3x^2 - 3\left[x \cdot 2y\frac{dy}{dx} + y^2 \cdot 1\right] + 0 = 0$$

and $$3\left[x^2\frac{dy}{dx} + y \cdot 2x\right] - 3y^2\frac{dy}{dx} - 0 = 0$$

$\Rightarrow$ $$3x \cdot 2y\frac{dy}{dx} + 3y^2 = 3x^2$$

and $$3y^2\frac{dy}{dx} = 3x^2\frac{dy}{dx} + 6xy$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{3x^2 - 3y^2}{6xy}$$

and $$\frac{dy}{dx} = \frac{6xy}{3y^2 - 3x^2}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right) = \frac{3(x^2 - y^2)}{6xy}$$

and $$\left(\frac{dy}{dx}\right) = \frac{-6xy}{3(x^2 - y^2)}$$

$\Rightarrow$ $$m_1 = \frac{(x^2 - y^2)}{2xy}$$

and $$m_2 = \frac{-2xy}{x^2 - y^2}$$

$\therefore$ $$m_1m_2 = \frac{x^2 - y^2}{2xy} \cdot \frac{-(2xy)}{x^2 - y^2} = -1$$

Hence, both the curves are intersecting at right angle *i.e.,* making $\frac{\pi}{2}$ with each other.

**Q. 46** The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is

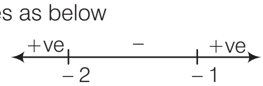
(a) $[-1, \infty)$ (b) $[-2, -1]$ (c) $(-\infty, -2]$ (d) $[-1, 1]$

**Sol.** ***(b)*** We have, $$f(x) = 2x^3 + 9x^2 + 12x - 1$$

$\therefore$ $$f'(x) = 6x^2 + 18x + 12$$
$$= 6(x^2 + 3x + 2) = 6(x + 2)(x + 1)$$

So, $f'(x) \le 0$, for decreasing.

On drawing number lines as below




We see that $f'(x)$ is decreasing in $[-2, -1]$.

**Q. 47** If $f : R \to R$ be defined by $f(x) = 2x + \cos x$, then $f$

(a) has a minimum at $x = \pi$ (b) has a maximum at $x = 0$

(c) is a decreasing function (d) is an increasing function

**Sol.** *(d)* We have, $f(x) = 2x + \cos x$

$\therefore$ $f'(x) = 2 + (-\sin x) = 2 - \sin x$

Since, $f'(x) > 0, \forall\ x$

Hence, $f(x)$ is an increasing function.

**Q. 48** If $y = x(x-3)^2$ decreases for the values of $x$ given by

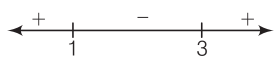
(a) $1 < x < 3$ (b) $x < 0$ (c) $x > 0$ (d) $0 < x < \dfrac{3}{2}$

**Sol.** *(a)* We have, $y = x(x-3)^2$

$$\therefore \quad \frac{dy}{dx} = x \cdot 2(x-3) \cdot 1 + (x-3)^2 \cdot 1$$

$$= 2x^2 - 6x + x^2 + 9 - 6x = 3x^2 - 12x + 9$$

$$= 3(x^2 - 3x - x + 3) = 3(x-3)(x-1)$$




So, $y = x(x-3)^2$ decreases for (1, 3).

[since, $y' < 0$ for all $x \in (1, 3)$, hence $y$ is decreasing on (1, 3)]

**Q. 49** The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly

(a) increasing in $\left(\pi, \dfrac{3\pi}{2}\right)$ (b) decreasing in $\left(\dfrac{\pi}{2}, \pi\right)$

(c) decreasing in $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ (d) decreasing in $\left[0, \dfrac{\pi}{2}\right]$

**Sol.** *(b)* We have, $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$

$\therefore$ $f'(x) = 12\sin^2 x \cdot \cos x - 12\sin x \cdot \cos x + 12\cos x$

$$= 12[\sin^2 x \cdot \cos x - \sin x \cdot \cos x + \cos x]$$

$$= 12\cos x[\sin^2 x - \sin x + 1]$$

$\Rightarrow$ $f'(x) = 12\cos x[\sin^2 x + (1 - \sin x)]$ ...(i)

$\because$ $1 - \sin x \ge 0$ and $\sin^2 x \ge 0$

$\therefore$ $\sin^2 x + 1 - \sin x \ge 0$

Hence, $f'(x) > 0$, when $\cos x > 0$ *i.e.,* $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

So, $f(x)$ is increasing when $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and $f'(x) < 0$, when $\cos x < 0$ *i.e.,* $x \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing when $x \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Since, $\left(\dfrac{\pi}{2}, \pi\right) \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing in $\left(\dfrac{\pi}{2}, \pi\right)$.

**Q. 50** Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$?

(a) $\sin 2x$ (b) $\tan x$ (c) $\cos x$ (d) $\cos 3x$

**Sol.** *(c)* In the interval $\left(0, \frac{\pi}{2}\right)$, $f(x) = \cos x$

$\Rightarrow$ $f'(x) = -\sin x$

which gives $f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

**Q. 51** The function $f(x) = \tan x - x$

(a) always increases
(b) always decreases
(c) never increases
(d) sometimes increases and sometimes decreases

**Sol.** *(a)* We have, $f(x) = \tan x - x$

$\therefore$ $f'(x) = \sec^2 x - 1$

$\Rightarrow$ $f'(x) > 0, \forall x \in R$

So, $f(x)$ always increases.

**Q. 52** If $x$ is real, then the minimum value of $x^2 - 8x + 17$ is

(a) $-1$ (b) 0 (c) 1 (d) 2

**Sol.** *(c)* Let $f(x) = x^2 - 8x + 17$

$\therefore$ $f'(x) = 2x - 8$

So, $f'(x) = 0$, gives $x = 4$

Now, $f''(x) = 2 > 0, \forall x$

So, $x = 4$ is the point of local minima.

$\therefore$ Minimum value of $f(x)$ at $x = 4$,

$$f(4) = 4 \times 4 - 8 \times 4 + 17 = 1$$

**Q. 53** The smallest value of polynomial $x^3 - 18x^2 + 96x$ in [0, 9] is

(a) 126 (b) 0 (c) 135 (d) 160

**Sol.** *(b)* We have, $f(x) = x^3 - 18x^2 + 96x$

$\therefore$ $f'(x) = 3x^2 - 36x + 96$

So, $f'(x) = 0$

Gives, $3x^2 - 36x + 96 = 0$

$\Rightarrow$ $3(x^2 - 12x + 32) = 0$

$\Rightarrow$ $(x - 8)(x - 4) = 0$

$\Rightarrow$ $x = 8, 4 \in [0, 9]$

We shall now evaluate the value of $f$ at these points and at the end points of the interval [0, 9] *i.e.,* at $x = 4$ and $x = 8$ and at $x = 0$ and at $x = 9$.

$$\therefore \quad f(4) = 4^3 - 18 \cdot 4^2 + 96 \cdot 4$$
$$= 64 - 288 + 384 = 160$$
$$f(8) = 8^3 - 18 \cdot 8^2 + 96 \cdot 8 = 128$$
$$f(9) = 9^3 - 18 \cdot 9^2 + 96 \cdot 9$$
$$= 729 - 1458 + 864 = 135$$
and
$$f(0) = 0^3 - 18 \cdot 0^2 + 96 \cdot 0 = 0$$

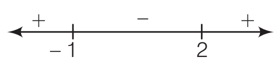
Thus, we conclude that absolute minimum value of $f$ on [0, 9] is 0 occurring at $x = 0$.

## Q. 54 The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(a) two points of local maximum
(b) two points of local minimum
(c) one maxima and one minima
(d) no maxima or minima

**Sol.** ***(c)*** We have
$$f(x) = 2x^3 - 3x^2 - 12x + 4$$
$$\therefore \quad f'(x) = 6x^2 - 6x - 12$$
Now,
$$f'(x) = 0 \Rightarrow 6(x^2 - x - 2) = 0$$
$$\Rightarrow \quad 6(x + 1)(x - 2) = 0$$
$$\Rightarrow \quad x = -1 \text{ and } x = +2$$

On number line for $f'(x)$, we get




Hence $x = -1$ is point of local maxima and $x = 2$ is point of local minima.
So, $f(x)$ has one maxima and one minima.

## Q. 55 The maximum value of $\sin x \cdot \cos x$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\sqrt{2}$ (d) $2\sqrt{2}$

**Sol.** ***(b)*** We have,
$$f(x) = \sin x \cdot \cos x = \frac{1}{2} \sin 2x$$
$$\therefore \quad f'(x) = \frac{1}{2} \cdot \cos 2x \cdot 2 = \cos 2x$$
Now,
$$f'(x) = 0 \Rightarrow \cos 2x = 0$$
$$\Rightarrow \quad \cos 2x = \cos \frac{\pi}{2} \Rightarrow x = \frac{\pi}{4}$$
Also
$$f''(x) = \frac{d}{dx} \cos 2x = -\sin 2x \cdot 2 = -2 \sin 2x$$
$$\therefore \quad [f''(x)]_{\text{at } x = \pi/4} = -2 \cdot \sin 2 \cdot \frac{\pi}{4} = -2 \sin \frac{\pi}{2} = -2 < 0$$

At $\frac{\pi}{4}$, $f(x)$ is maximum and $\frac{\pi}{4}$ is point of maxima.

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot \sin 2 \cdot \frac{\pi}{4} = \frac{1}{2}$$

**Q. 56** At $x = \frac{5\pi}{6}$, $f(x) = 2\sin 3x + 3\cos 3x$ is

(a) maximum (b) minimum
(c) zero (d) neither maximum nor minimum

**Sol.** ***(d)*** We have,
$$f(x) = 2\sin 3x + 3\cos 3x$$
$$\therefore \quad f'(x) = 2\cdot\cos 3x \cdot 3 + 3(-\sin 3x)\cdot 3$$
$$\Rightarrow \quad f'(x) = 6\cos 3x - 9\sin 3x \quad \text{...(i)}$$
Now,
$$f''(x) = -18\sin 3x - 27\cos 3x$$
$$= -9(2\sin 3x + 3\cos 3x)$$
$$\therefore \quad f'\left(\frac{5\pi}{6}\right) = 6\cos\left(3\cdot\frac{5\pi}{6}\right) - 9\sin\left(3\cdot\frac{5\pi}{6}\right)$$
$$= 6\cos\frac{5\pi}{2} - 9\sin\frac{5\pi}{2}$$
$$= 6\cos\left(2\pi + \frac{\pi}{2}\right) - 9\sin\left(2\pi + \frac{\pi}{2}\right)$$
$$= 0 - 9 \neq 0$$
So, $x = \frac{5\pi}{6}$ cannot be point of maxima or minima.

Hence, $f(x)$ at $x = \frac{5\pi}{6}$ is neither maximum nor minimum.

**Q. 57** The maximum slope of curve $y = -x^3 + 3x^2 + 9x - 27$ is

(a) 0 (b) 12 (c) 16 (d) 32

**Sol.** ***(b)*** We have,
$$y = -x^3 + 3x^2 + 9x - 27$$
$$\therefore \quad \frac{dy}{dx} = -3x^2 + 6x + 9 = \text{Slope of the curve}$$
and
$$\frac{d^2y}{dx^2} = -6x + 6 = -6(x-1)$$
$$\therefore \quad \frac{d^2y}{dx^2} = 0$$
$$\Rightarrow \quad -6(x-1) = 0 \Rightarrow x = 1 > 0$$
Now,
$$\frac{d^3y}{dx^3} = -6 < 0$$
So, the maximum slope of given curve is at $x = 1$.
$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x=1)} = -3\cdot 1^2 + 6\cdot 1 + 9 = 12$$

**Q. 58** The functin $f(x) = x^x$ has a stationary point at

(a) $x = e$ (b) $x = \frac{1}{e}$ (c) $x = 1$ (d) $x = \sqrt{e}$

**Sol.** ***(b)*** We have,
$$f(x) = x^x$$
Let
$$y = x^x$$
and
$$\log y = x\log x$$
$$\therefore \quad \frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x\cdot 1$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + \log x) \cdot x^x$$

$$\therefore \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1 + \log x) \cdot x^x = 0$$

$$\Rightarrow \quad \log x = -1$$

$$\Rightarrow \quad \log x = \log e^{-1}$$

$$\Rightarrow \quad x = e^{-1}$$

$$\Rightarrow \quad x = \frac{1}{e}$$

Hence, $f(x)$ has a stationary point at $x = \frac{1}{e}$.

**Q. 59** The maximum value of $\left(\frac{1}{x}\right)^x$ is

(a) $e$ (b) $e^e$ (c) $e^{1/e}$ (d) $\left(\frac{1}{e}\right)^{1/e}$

**Sol.** *(c)* Let

$$y = \left(\frac{1}{x}\right)^x$$

$$\Rightarrow \quad \log y = x \cdot \log \frac{1}{x}$$

$$\therefore \quad \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) + \log \frac{1}{x} \cdot 1$$

$$= -1 + \log \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \left(\log \frac{1}{x} - 1\right) \cdot \left(\frac{1}{x}\right)^x$$

Now, $$\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \log \frac{1}{x} = 1 = \log e$$

$$\Rightarrow \quad \frac{1}{x} = e$$

$$\therefore \quad x = \frac{1}{e}$$

Hence, the maximum value of $f\left(\frac{1}{e}\right) = (e)^{1/e}$.

## Fillers

**Q. 60** The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point ......... .

**Sol.** The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point (3, 34).

Given, equation of curves are $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$

$$\therefore \quad \frac{dy}{dx} = 8x + 2$$

and $$\frac{dy}{dx} = 3x^2 - 1$$

So, the slope of both curves should be same

$$\therefore \quad 8x + 2 = 3x^2 - 1$$

$$\Rightarrow \quad 3x^2 - 8x - 3 = 0$$

$$\Rightarrow \quad 3x^2 - 9x + x - 3 = 0$$

$$\Rightarrow \quad 3x(x - 3) + 1(x - 3) = 0$$

$$\Rightarrow \quad (3x + 1)(x - 3) = 0$$

$$\therefore \quad x = -\frac{1}{3} \text{ and } x = 3,$$

For $x = -\frac{1}{3}$,
$$y = 4\cdot\left(-\frac{1}{3}\right)^2 + 2\cdot\left(\frac{-1}{3}\right) - 8$$

$$= \frac{4}{9} - \frac{2}{3} - 8 = \frac{4 - 6 - 72}{9}$$

$$= -\frac{74}{9}$$

and for $x = 3$, $y = 4\cdot(3)^2 + 2\cdot(3) - 8$

$$= 36 + 6 - 8 = 34$$

So, the required points are (3, 34) and $\left(-\frac{1}{3}, \frac{-74}{9}\right)$.

**Q. 61** The equation of normal to the curve $y = \tan x$ at (0, 0) is ......... .

**Sol.** The equation of normal to the curve $y = \tan x$ at (0, 0) is $x + y = 0$.

$$\because \quad y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,0)} = \sec^2 0 = 1 \text{ and } -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{1}{1}$$

$\therefore$ Equation of normal to the curve $y = \tan x$ at (0, 0) is

$$y - 0 = -1(x - 0)$$

$$\Rightarrow \quad y + x = 0$$

**Q. 62** The values of $a$ for which the function $f(x) = \sin x - ax + b$ increases on $R$ are ......... .

**Sol.** The values of $a$ for which the function $f(x) = \sin x - ax + b$ increases on $R$ are $(-\infty, -1)$.

$\because \quad f'(x) = \cos x - a$

and $\quad f'(x) > 0 \Rightarrow \cos x > a$

Since, $\quad \cos x \in [-1, 1]$

$\Rightarrow \quad a < -1 \Rightarrow a \in (-\infty, -1)$

**Q. 63** The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, (where, $x > 0$) decreases in the interval ......... .

**Sol.** The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, where $x > 0$, decreases in the interval $(1, \infty)$.

$$\because \quad f'(x) = \frac{x^4 \cdot 4x - (2x^2 - 1) \cdot 4x^3}{x^8} = \frac{4x^5 - 8x^5 + 4x^3}{x^8}$$

$$= \frac{-4x^5 + 4x^3}{x^8} = \frac{4x^3(-x^2 + 1)}{x^8}$$

Also, $\quad f'(x) < 0$

$$\Rightarrow \quad \frac{4x^3(1 - x^2)}{x^8} < 0 \Rightarrow x^2 > 1$$

$\Rightarrow \quad x > \pm 1$

$\therefore \quad x \in (1, \infty)$

**Q. 64** The least value of function $f(x) = ax + \dfrac{b}{x}$ (where, $a > 0$, $b > 0$, $x > 0$) is ......... .

**Sol.** The least value of function $f(x) = ax + \dfrac{b}{x}$ (where, $a > 0, b > 0, x > 0$) is $2\sqrt{ab}$.

$$\because \quad f'(x) = a - \frac{b}{x^2} \text{ and } f'(x) = 0$$

$$\Rightarrow \quad a = \frac{b}{x^2}$$

$$\Rightarrow \quad x^2 = \frac{b}{a} \Rightarrow x = \pm\sqrt{\frac{b}{a}}$$

Now, $$f''(x) = -b \cdot \frac{(-2)}{x^3} = +\frac{2b}{x^3}$$

At $x = \sqrt{\dfrac{b}{a}}$, $$f''(x) = +\frac{2b}{\left(\dfrac{b}{a}\right)^{3/2}} = \frac{+2b \cdot a^{3/2}}{b^{3/2}}$$

$$= +2b^{-1/2} \cdot a^{3/2} = +2\sqrt{\frac{a^3}{b}} > 0 \qquad [\because a, b > 0]$$

$\therefore$ Least value of $f(x)$, $$f\left(\sqrt{\frac{b}{a}}\right) = a \cdot \sqrt{\frac{b}{a}} + \frac{b}{\sqrt{\dfrac{b}{a}}}$$

$$= a \cdot a^{-1/2} \cdot b^{1/2} + b \cdot b^{-1/2} \cdot a^{1/2}$$

$$= \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$