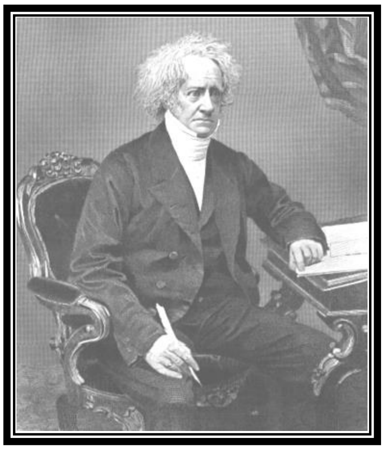
Chapter

5

Inverse Trigonometrical Functions

CONTENTS



Sir John F.W. Hersehel

Daniel Bernoulli was the first to use symbols for the inverse trigonometric functions. In 1729 he used A S. for arcsine in Comment. acad. sc. Petrop., Vol. II.

According to Cajori (vol. 2, page 176) the inverse trigonometric function notation utilizing the exponent^{-1} was introduced by Sir John Frederick William Herschel in 1813 in the Philosophical Transactions of London.The symbols $\sin^{-1}x$, $\cos^{-1}x$ etc. for arc sinx, arc cosx etc. were also suggested by the astronomer Sir John F.W. Hersehel.

In American and English books the symbol $\sin^{-1}y$ is generally used; on the continent of Europe the symbol arc sin y is the one that is met.

Inverse Trigonometrical Functions

The inverse of a function $f: A \to B$ exists if f is one-one onto *i.e.*, a bijection and is given by $f(x) = y \Rightarrow f^{-1}(y) = x$.

Consider the sine function with domain R and range [–1, 1]. Clearly this function is not a bijection and so it is not invertible. If we restrict the domain of it in such a way that it becomes one–one, then it would become invertible. If we consider sine as a function with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and co-domain [–1, 1], then it is a bijection and therefore, invertible. The inverse of sine function is defined as $\sin^{-1} x = \theta \Leftrightarrow \sin\theta = x$, where $\theta \leftarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x \leftarrow [-1, 1]$.

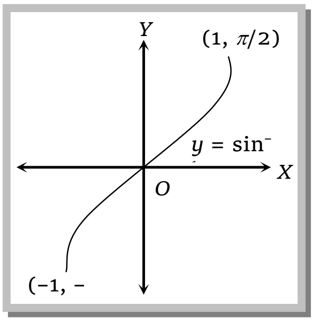
5.1 Properties of Inverse Trigonometric Functions

(1) **Meaning of inverse function**

(i) $\sin\theta = x \Rightarrow \sin^{-1} x = \theta$ (ii) $\cos\theta = x \Rightarrow \cos^{-1} x = \theta$ (iii) $\tan\theta = x \Rightarrow \tan^{-1} x = \theta$

(iv) $\cot\theta = x \Rightarrow \cot^{-1} x = \theta$ (v) $\sec\theta = x \Rightarrow \sec^{-1} x = \theta$ (vi) $\operatorname{cosec}\theta = x \Rightarrow \operatorname{cosec}^{-1} x = \theta$

(2) **Domain and range of inverse functions**



(i) If $\sin y = x$, then $y = \sin^{-1} x$, under certain condition.

$-1 \le \sin y \le 1$; but $\sin y = x$. $\therefore -1 \le x \le 1$

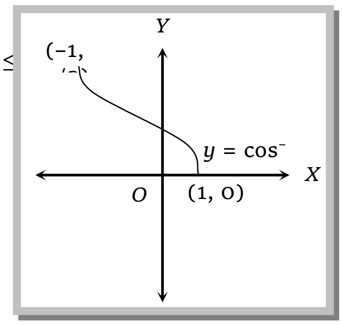
Again, $\sin y = -1 \Rightarrow y = -\frac{\pi}{2}$ and $\sin y = 1 \Rightarrow y = \frac{\pi}{2}$.

Keeping in mind numerically smallest angles or real numbers. $\therefore -\frac{\pi}{2}$

These restrictions on the values of x and y provide us with the domain and range for the function $y = \sin^{-1} x$.

i.e., Domain : $x \in [-1, 1]$

Range: $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



(ii) Let $\cos y = x$, then $y = \cos^{-1} x$, under certain conditions $-1 \le$

$\Rightarrow$ $-1 \le x \le 1$

$\cos y = -1 \Rightarrow y = \pi$

$\cos y = 1 \Rightarrow y = 0$

$\therefore$ $0 \le y \le \pi$ {as cos x is a decreasing function in $[0, \pi]$;

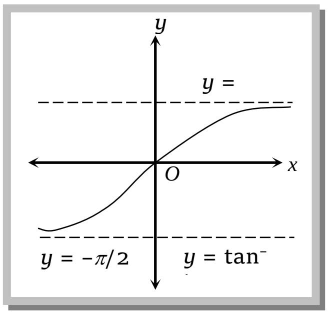
hence $\cos\pi \le \cos y \le \cos 0$

These restrictions on the values of x and y provide us the domain and range for the function $y = \cos^{-1} x$.

i.e. Domain: $x \in [-1, 1]$

Range : $y \in [0, \pi]$

(iii) If $\tan y = x$, then $y = \tan^{-1} x$, under certain conditions.

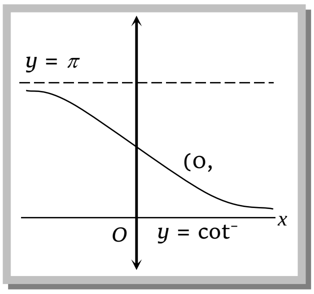


Here, $\tan y \in R \Rightarrow x \in R$, $-\infty < \tan y < \infty \Rightarrow -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

Thus, Domain $x \in R$;

Range $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(iv) If $\cot y = x$, then $y = \cot^{-1} x$



under certain conditions, $\cot y \in R \Rightarrow x \in R$;

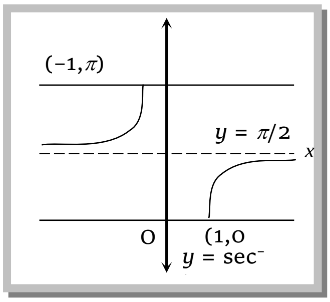
$-\infty < \cot y < \infty \Rightarrow 0 < y < \pi$

These conditions on x and y make the function, $\cot y = x$ one-one and onto so that the inverse function exists. *i.e.*, $y = \cot^{-1} x$ is meaningful.

$\Rightarrow$ Domain : $x \in R$

Range : $y \in (0, \pi)$

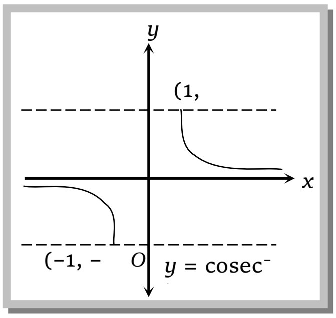
(v) If $\sec y = x$, then $y = \sec^{-1} x$, where $|x| \geq 1$ and $0 \leq y \leq \pi, y \neq \dfrac{\pi}{2}$



Here, Domain: $x \in R - (-1, 1)$

Range: $y \in [0, \pi] - \left\{\dfrac{\pi}{2}\right\}$

(vi) If $\operatorname{cosec} y = x$, then $y = \operatorname{cosec}^{-1} x$



Where $|x| \geq 1$ and $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}, y \neq 0$

Here, Domain $\in R - (-1, 1)$

Range $\in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$

Function	Domain (*D*)	Range (*R*)
$\sin^{-1} x$	$-1 \leq x \leq 1$ or $[-1, 1]$	$-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$ or $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
$\cos^{-1} x$	$-1 \leq x \leq 1$ or $[-1, 1]$	$0 \leq \theta \leq \pi$ or $[0, \pi]$
$\tan^{-1} x$	$-\infty < x < \infty$ *i.e.*, $x \in R$ or $(-\infty, \infty)$	$-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ or $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
$\cot^{-1} x$	$-\infty < x < \infty$ *i.e.*, $x \in R$ or	$0 < \theta < \pi$ or $(0, \pi)$

	$(-\infty, \infty)$	
$\sec^{-1} x$	$x \le -1, x \ge 1$ or $(-\infty, -1] \cup [1, \infty)$	$\theta \ne \frac{\pi}{2}, 0 \le \theta \le \pi$ or $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\text{cosec}^{-1} x$	$x \le -1, x \ge 1$ or $(-\infty, -1] \cup [1, \infty)$	$\theta \ne 0, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

(3) $\sin^{-1}(\sin\theta) = \theta$, Provided that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos^{-1}(\cos\theta) = \theta$, Provided that $0 \le \theta \le \pi$

$\tan^{-1}(\tan\theta) = \theta$, Provided that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cot^{-1}(\cot\theta) = \theta$, Provided that $0 < \theta < \pi$

$\sec^{-1}(\sec\theta) = \theta$, Provided that $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

$\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, Provided that $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$

(4) $\sin(\sin^{-1} x) = x$, Provided that $-1 \le x \le 1$, $\cos(\cos^{-1} x) = x$, Provided that $-1 \le x \le 1$

tan $(\tan^{-1} x) = x$, Provided that $-\infty < x < \infty$ $\cot(\cot^{-1} x) = x$, Provided that $-\infty < x < \infty$

$\sec(\sec^{-1} x) = x$, Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

$\text{cosec}\,(\text{cosec}^{-1} x) = x$, Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

(5) $\sin^{-1}(-x) = -\sin^{-1} x$ $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$

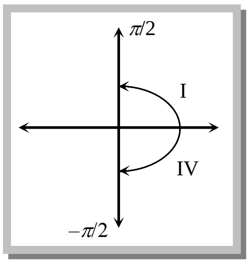
$\cot^{-1}(-x) = \pi - \cot^{-1} x$ $\sec^{-1}(-x) = \pi - \sec^{-1} x$ $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$

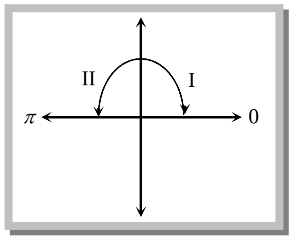
(6) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for all $x \in [-1, 1]$ $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, for all $x \in R$

$\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Important Tips

- *Here;* $\sin^{-1} x, \text{cosec}^{-1} x, \tan^{-1} x$ *belong to I and IV Quadrant.*
- *Here;* $\cos^{-1} x, \sec^{-1} x, \cot^{-1} x$ *belong to I and II Quadrant.*
- *I Quadrant is common to all the inverse functions.*
- *III Quadrant is not used in inverse function.*
- *IV Quadrant is used in the clockwise direction i.e.,* $-\frac{\pi}{2} \le y \le 0$





(7) **Principal values for inverse circular functions**

Principal values for $x \ge 0$	Principal values for $x < 0$
$0 \le \sin^{-1} x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \sin^{-1} x < 0$
$0 \le \cos^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1} x \le \pi$

$0 \le \tan^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1} x < 0$
$0 < \cot^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1} x < \pi$
$0 \le \sec^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1} x \le \pi$
$0 < \text{cosec}^{-1} x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \text{cosec}^{-1} x < 0$

Thus $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, not $\frac{5\pi}{6}$; $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ not $\frac{4\pi}{3}$; $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ not $\frac{2\pi}{3}$; $\cot^{-1}(-1) = \frac{3\pi}{4}$ not $-\frac{\pi}{4}$ etc.

Note : ❑ $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$ are also written as arc $\sin x$, arc $\cos x$ and arc $\tan x$ respectively.

❑ It should be noted that if not otherwise stated only principal values of inverse circular functions are to be considered.

(8) **Conversion property :** Let, $\sin^{-1} x = y \Rightarrow x = \sin y \Rightarrow \text{cosec } y = \left(\frac{1}{x}\right) \Rightarrow y = \text{cosec}^{-1}\left(\frac{1}{x}\right)$

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\left(\frac{1}{x}\right)$$

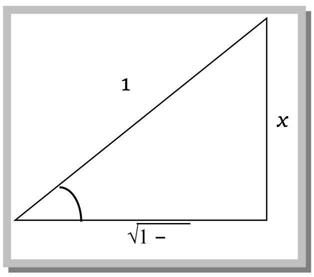
$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\frac{1}{x} = \text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

Note : ❑ $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

❑ $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

❑ $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$



(9) **General values of inverse circular functions:** We know that if α is the smallest angle whose sine is x, then all the angles whose sine is x can be written as $nx + (-1)^n \alpha$, where $n = 0, 1, 2, ...$ Therefore, the general value of $\sin^{-1} x$ can be taken as $n\pi + (-1)^n \alpha$. The general value of $\sin^{-1} x$ is denoted by $\sin^{-1} x$.

Thus, we have $\boxed{\sin^{-1} x = n\pi + (-1)^n \alpha, -1 \le x \le 1, \text{ if } \sin\alpha = x \text{ and } -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}}$

Similarly, general values of other inverse circular functions are given as follows:

$\cos^{-1} x = 2n\pi \pm \alpha, -1 \le x \le 1$;	If $\cos\alpha = x$, $0 \le \alpha \le \pi$
$\tan^{-1} x = n\pi + \alpha, \quad x \in R$;	If $\tan\alpha = x, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$
$\cot^{-1} x = n\pi + \alpha, \quad x \in R$;	If $\cot\alpha = x$, $0 < \alpha < \pi$
$\sec^{-1} x = 2n\pi \pm \alpha$, $x \ge 1$ or $x \le -1$;	If $\sec\alpha = x, 0 \le \alpha \le \pi$ and $\ne \frac{\pi}{2}$
$\text{cosec}^{-1} x = n\pi + (-1)^n \alpha$, $x \ge 1$ or $x \le -1$;	If $\text{cosec}\,\alpha = x$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ and $x \ne 0$

Example: 1 The principal value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is **[Roorkee 1992]**

(a) $-\frac{2\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{5\pi}{8}$

Solution: (b) $\sin^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}$ $\left(\because -\frac{\pi}{2} < \sin^{-1} x < \frac{\pi}{2}\right)$

Example: 2 $\sec^{-1}[\sec(-30^o)] =$ **[MP PET 1992]**

(a) -60^o (b) -30^o (c) 30^o (d) 150^o

Solution: (c) $\sec^{-1}[\sec(-30^o)] = \sec^{-1}(\sec 30^o) = 30^o$.

Example: 3 The principal value of $\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is **[MP PET 1996]**

(a) $\frac{5\pi}{3}$ (b) $-\frac{5\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $\frac{4\pi}{3}$

Solution: (c) $\sin^{-1}\left(\sin\frac{5\pi}{3}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

Example: 4 The principal value of $\sin^{-1}\left[\sin\left(\frac{2\pi}{3}\right)\right]$ is **[IIT 1986]**

(a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) None of these

Solution: (d) The principal vlaue of $\sin^{-1}[\sin(\pi - \frac{2\pi}{3})]$ = $\sin^{-1}\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$.

Example: 5 Considering only the principal values, if $\tan(\cos^{-1} x) = \sin\left[\cot^{-1}\left(\frac{1}{2}\right)\right]$, then x is equal to

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{2}{\sqrt{5}}$ (c) $\frac{3}{\sqrt{5}}$ (d) $\frac{\sqrt{5}}{3}$

Solution: (d) Put $\cot^{-1}\left(\frac{1}{2}\right) = \theta \Rightarrow \cot\theta = \frac{1}{2}$

$\therefore \sin\theta = \frac{2}{\sqrt{5}}$. Put $\cos^{-1} x = \phi$ then $x = \cos\phi$

Also $\because \tan\phi = \frac{2}{\sqrt{5}}$, $\therefore x = \cos\phi = \frac{\sqrt{5}}{3}$.

Example: 6 If $\theta = \sin^{-1}[\sin(-600°)]$, then one of the possible value of θ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $-\frac{2\pi}{3}$

Solution: (a) $\theta = \sin^{-1}[\sin(-600^o)] = \sin^{-1}[-\sin(360+240)]$

$\Rightarrow \theta = \sin^{-1}[-\sin 240^o] = \sin^{-1}[-\sin(180^o+60^o)] \Rightarrow \theta = \sin^{-1}\sin 60^o = \sin^{-1}\left[\sin\left(\frac{\pi}{3}\right)\right] = \frac{\pi}{3}$.

Example: 7 Value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is **[Roorkee 2000]**

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{10\pi}{3}$

Solution: (a) $\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\sin\frac{5\pi}{3}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{\pi}{3}\right)\right] + \sin^{-1}\left[\sin\left(2\pi - \frac{\pi}{3}\right)\right] = \frac{\pi}{3} - \frac{\pi}{3} = 0$.

Example: 8 The equation $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6}$ has **[AMU 1999]**

(a) No solution (b) Only one solution (c) Two solutions (d) Three solutions

Solution: (a) Given equation is $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6} \Rightarrow \cos^{-1}x + (\cos^{-1}x + \sin^{-1}x) = \frac{11\pi}{6}$

$\Rightarrow \cos^{-1}x = \frac{11\pi}{6} - \frac{\pi}{2} \Rightarrow \cos^{-1}x = \frac{4\pi}{3}$, which is not possible as $\cos^{-1}x \in [0,\pi]$.

Example: 9 If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y =$ **[EAMCET 1994]**

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) π

Solution: (b) $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$

$\Rightarrow \frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}y = \frac{2\pi}{3} \Rightarrow \cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$.

Example: 10 If $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x$, $x \ge 0$ then the smallest interval in which θ lies is

(a) $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$ (b) $0 \le \theta \le \frac{\pi}{4}$ (c) $-\frac{\pi}{4} \le \theta \le 0$ (d) $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

Solution: (b) $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

We know $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} > -\tan^{-1}x > -\frac{\pi}{2}$ $\therefore 0 < \frac{\pi}{2} - \tan^{-1}x < \frac{\pi}{4}$.

Example: 11 If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - ...\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ...\right) = \frac{\pi}{2}$ for $0 < |x| < \sqrt{2}$, then x equals **[IIT Screening 2001]**

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) – 1

Solution: (b) We know that $\sin^{-1}y + \cos^{-1}y = \frac{\pi}{2}, |y| \le 1$

$\therefore$ According to question, $x - \frac{x^2}{2} + \frac{x^3}{4} - .... = x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ....$ ...

$\Rightarrow \frac{x}{1+\frac{x}{2}} = \frac{x^2}{1+\frac{x^2}{2}}$, $(\because 0 < |x| < \sqrt{2}) \Rightarrow \frac{x}{2+x} = \frac{x^2}{2+x^2} \Rightarrow 2x + x^3 = 2x^2 + x^3 \Rightarrow x = x^2$

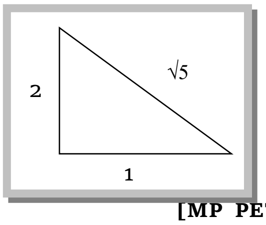
$\therefore x - x^2 = 0 \Rightarrow x(1-x) = 0 \Rightarrow x = 0$ and $x = 1$, but $x \neq 0$. So, $x = 1$.

Example: 12 If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is

[Karnataka 1999; Roorkee 1999]

(a) 0 (b) $\frac{1}{\sqrt{5}}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{3}}{2}$

Solution: (b) $\sin^{-1} x + \cot^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{2}$ $\left(\because \cot^{-1}\frac{1}{2} = \cos^{-1}\frac{1}{\sqrt{5}}\right)$



$\sin^{-1} x + \cos^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{2}$; Clearly, $x = \frac{1}{\sqrt{5}}$.

Example: 13 The value of $\sin(\cot^{-1} x)$ is **[MP PET 2001; UPSEAT 1987]**

(a) $(1+x^2)^{3/2}$ (b) $(1+x^2)^{-3/2}$ (c) $(1+x^2)^{1/2}$ (d) $(1+x^2)^{-1/2}$

Solution: (d) $\sin(\cot^{-1} x) = \sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$.

Example: 14 The number of real solutions of $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is **[IIT Screening1999]**

(a) Zero (b) One (c) Two (d) Infinite

Solution: (c) $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

$\tan^{-1}\sqrt{x(x+1)}$ is defined, when $x(x+1) \geq 0$ ....(i)

$\sin^{-1}\sqrt{x^2+x+1}$ is defined, when $0 \leq x(x+1)+1 \leq 1$ or $0 \leq x(x+1) \leq 0$ ....(ii)

From (i) and (ii), $x(x+1) = 0$ or $x = 0$ and –1.

Hence, number of solutions is 2.

5.2 Formulae for Sum and Difference of Inverse Trigonometric Function

(1) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; If $x > 0, y > 0$ and $xy < 1$

(2) $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; If $x > 0, y > 0$ and $xy > 1$

(3) $\tan^{-1} x + \tan^{-1} y = -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; If $x < 0, y < 0$ and $xy > 1$

(4) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$; If $xy > -1$

(5) $\tan^{-1} - \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right)$; If $x > 0, y < 0$ and $xy < -1$

(6) $\tan^{-1} x - \tan^{-1} y = -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right)$; If $x < 0, y > 0$ and $xy < -1$

(7) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

(8) $\tan^{-1}x_1+\tan^{-1}x_2+.........+\tan^{-1}x_n=\tan^{-1}\left[\frac{S_1-S_3+S_5-..........}{1-S_2+S_4-S_6+........}\right]$,

where S_k denotes the sum of the products of $x_1,x_2,........,x_n$ taken k at a time.

(9) $\cot^{-1}x+\cot^{-1}y=\cot^{-1}\frac{xy-1}{y+x}$

(10) $\cot^{-1}x-\cot^{-1}y=\cot^{-1}\frac{xy+1}{y-x}$

(11) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$;

If $-1\le x,y\le 1$ and $x^2+y^2\le 1$ or if $xy<0$ and $x^2+y^2>1$

(12) $\sin^{-1}x+\sin^{-1}y=\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$, If $0<x,y\le 1$ and $x^2+y^2>1$

(13) $\sin^{-1}x+\sin^{-1}y=-\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$, If $-1\le x;y<0$ and $x^2+y^2>1$

(14) $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}$, If $-1\le x;y\le 1$ and $x^2+y^2\le 1$ if or $xy>0$ and $x^2+y^2>1$.

(15) $\sin^{-1}x-\sin^{-1}y=\pi-\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}$, If $0<x\le 1,-1\le y<0$ and $x^2+y^2>1$.

(16) $\sin^{-1}x-\sin^{-1}y=-\pi-\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}$, If $-1\le x<0,0<y\le 1$ and $x^2+y^2>1$.

(17) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\{xy-\sqrt{1-x^2}.\sqrt{1-y^2}\}$, If $-1\le x,y\le 1$ and $x+y\ge 0$.

(18) $\cos^{-1}x+\cos^{-1}y=2\pi-\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}$, If $-1\le x,y\le 1$ and $x+y\le 0$

(19) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}$, If $-1\le x,y\le 1$, and $x\le y$.

(20) $\cos^{-1}x-\cos^{-1}y=-\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}$, If $-1\le y\le 0$, $0<x\le 1$ and $x\ge y$.

Important Tips

☞ If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then $xy+yz+zx=1$.

☞ If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $x+y+z=xyz$.

☞ If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{\pi}{2}$, then $x^2+y^2+z^2+2xyz=1$.

☞ If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$.

☞ If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then $xy+yz+zx=3$.

- If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then $x^2+y^2+z^2+2xyz=1$.
- If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then $xy+yz+zx=3$.
- If $\sin^{-1}x+\sin^{-1}y=\theta$, then $\cos^{-1}x+\cos^{-1}y=\pi-\theta$.
- If $\cos^{-1}x+\cos^{-1}y=\theta$, then $\sin^{-1}x+\sin^{-1}y=\pi-\theta$.
- If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{2}$, then $xy=1$.
- If $\cot^{-1}x+\cot^{-1}y=\frac{\pi}{2}$, then $xy=1$.
- If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\theta$, then $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=\sin^2\theta$.

Example: 15 The value of $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right]$ is **[AMU 2001]**

(a) $\frac{6}{17}$ (b) $\frac{6}{\sqrt{13}}$ (c) $\frac{\sqrt{13}}{5}$ (d) $\frac{17}{6}$

Solution: (d) $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right] = \tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$

$$= \tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}.\frac{2}{3}}\right) = \tan\left[\tan^{-1}\frac{17}{12}\times\frac{12}{6}\right] = \frac{17}{6}.$$

Example: 16 $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=$ **[MP PET 1997, 2003; UPSEAT 2003; Karnataka CET 2001]**

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

Solution: (b) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}}=\tan^{-1}1=\frac{\pi}{4}$.

Example: 17 If $\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{2}{3}=\sin^{-1}x$, then x is equal to **[Roorkee 1995]**

(a) 0 (b) $\frac{\sqrt{5}-4\sqrt{2}}{9}$ (c) $\frac{\sqrt{5}+4\sqrt{2}}{9}$ (d) $\frac{\pi}{2}$

Solution: (c) $\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{2}{3}=\sin^{-1}\left[\frac{1}{3}\sqrt{1-\frac{4}{9}}+\frac{2}{3}\sqrt{1-\frac{1}{9}}\right]=\sin^{-1}\left[\frac{\sqrt{5}+4\sqrt{2}}{9}\right]$

Therefore, $x=\frac{\sqrt{5}+4\sqrt{2}}{9}$.

Example: 18 $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3$ is equal to **[MP PET 1993; Karnataka CET 1995]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution: (b) $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=\cot^{-1}\left(\frac{\sqrt{1-\frac{1}{5}}}{\frac{1}{\sqrt{5}}}\right)+\cot^{-1}3=\cot^{-1}(2)+\cot^{-1}(3)=\cot^{-1}\left(\frac{2\times 3-1}{3+2}\right)=\cot^{-1}(1)=\frac{\pi}{4}$.

Example: 19 If $\sin^{-1}\frac{3}{5}+\cos^{-1}\left(\frac{12}{13}\right)=\sin^{-1}C$, then $C=$ **[Pb. CET 1999]**

(a) $\frac{65}{56}$ (b) $\frac{24}{65}$ (c) $\frac{16}{65}$ (d) $\frac{56}{65}$

Solution: (d) Given, $\sin^{-1}C=\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}$

$\therefore\ \sin^{-1}C=\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}=\sin^{-1}\left\{\frac{3}{5}\sqrt{1-\frac{25}{169}}+\frac{5}{13}\sqrt{1-\frac{9}{25}}\right\}=\sin^{-1}\left(\frac{56}{65}\right)\Rightarrow C=\frac{56}{65}$.

Example: 20 If $f(x)=\cos^{-1}x+\cos^{-1}\left\{\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right\}$, then

(a) $f\left(\frac{2}{3}\right)=\frac{\pi}{3}$ (b) $f\left(\frac{2}{3}\right)=2\cos^{-1}\frac{2}{3}-\frac{\pi}{3}$ (c) $f\left(\frac{1}{3}\right)=\frac{\pi}{3}$ (d) $f\left(\frac{1}{3}\right)=2\cos^{-1}\frac{1}{3}-\frac{\pi}{3}$

Soltuion: (a,d) $f(x)=\cos^{-1}x+\cos^{-1}\left\{\frac{1}{2}x+\frac{\sqrt{3}}{2}.\sqrt{1-x^2}\right\}$

$=\cos^{-1}x\pm(\cos^{-1}\frac{1}{2}-\cos^{-1}x)$, according as $\cos^{-1}\frac{1}{2}>$ or $<\cos^{-1}x$

$=\cos^{-1}\frac{1}{2}$ if $\cos^{-1}\frac{1}{2}>\cos^{-1}x$, which holds for $x=\frac{2}{3}$

$=2\cos^{-1}x-\cos^{-1}\frac{1}{2}$ if $\cos^{-1}\frac{1}{2}<\cos^{-1}x$, which holds for $x=\frac{1}{3}$.

Example: 21 $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=$

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{2}$

Solution: (c) $\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{16}=\pi+\tan^{-1}\frac{48+15}{20-36}+\tan^{-1}\frac{63}{16}\ (xy>1)=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16}=\pi$.

Example: 22 If $\alpha=\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{1}{3}$ and $\beta=\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{1}{3}$, then

(a) $\alpha<\beta$ (b) $\alpha=\beta$ (c) $\alpha>\beta$ (d) None of these

Solution: (a) $\alpha=\sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{1}{9}}+\frac{1}{3}\sqrt{1-\frac{16}{25}}\right]=\sin^{-1}\left[\frac{8\sqrt{2}}{15}+\frac{3}{15}\right]=\sin^{-1}\left(\frac{8\sqrt{2}+3}{15}\right)$

Since $\frac{8\sqrt{2}+3}{15}<1,\therefore\alpha<\frac{\pi}{2}$

$\beta=\left(\frac{\pi}{2}-\sin^{-1}\frac{4}{5}+\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)=(\pi-\alpha)>\frac{\pi}{2}\Rightarrow\ \alpha<\beta$.

Example: 23 If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$, then $9x^2-12xy\cos\theta+4y^2=$

(a) $36\sin^2\theta$ (b) $36\cos^2\theta$ (c) $36\tan^2\theta$ (d) None of these

Solution: (a) $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$

$\Rightarrow \frac{x}{2}.\frac{y}{3}-\sqrt{\left(1-\frac{x^2}{4}\right)}\sqrt{\left(1-\frac{y^2}{9}\right)}=\cos\theta$

$\therefore\ (xy-6\cos\theta)^2=(4-x^2)(9-y^2)\Rightarrow 9x^2-12xy\cos\theta+4y^2=36(1-\cos^2\theta)=36\sin^2\theta$.

Example: 24 The number of solutions of $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$ is

(a) 0 (b) 1 (c) 2 (d) Infinitie

Solution: (b) $\sin^{-1}2x=\sin^{-1}\frac{\sqrt{3}}{2}-\sin^{-1}x=\sin^{-1}\left[\frac{\sqrt{3}}{2}.\sqrt{1-x^2}-x\sqrt{1-\frac{3}{4}}\right]$

$\therefore\ 2x=\frac{\sqrt{3}}{2}\sqrt{1-x^2}-\frac{x}{2}$

$\therefore\left(\frac{5x}{2}\right)^2=\frac{3}{4}(1-x^2)$ or $28x^2=3\Rightarrow x=\sqrt{\frac{3}{28}}=\frac{1}{2}\sqrt{\frac{3}{7}}$, (not $-\frac{1}{2}\sqrt{\frac{3}{7}}$).

Example: 25 If $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\alpha$, then $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=$ **[UPSEAT 1999; MP PET 1995]**

(a) $\sin^2\alpha$ (b) $\cos^2\alpha$ (c) $\tan^2\alpha$ (d) $\cot^2\alpha$

Solution: (a) We have $\cos^{-1}\left[\frac{x}{a}.\frac{y}{b}-\sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)}\right]=\alpha\ \Rightarrow\frac{xy}{ab}-\sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)}=\cos\alpha$

$\therefore\ \left(\frac{xy}{ab}-\cos\alpha\right)^2=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$

$\frac{x^2y^2}{a^2b^2}+\cos^2\alpha-\frac{2xy}{ab}\cos\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}\ \Rightarrow\ \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=1-\cos^2\alpha=\sin^2\alpha$.

Example: 26 If a, b, c be positive real numbers and the value of $\theta=\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}$ $+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$, then $\tan\theta$ is

(a) 0 (b) 1 (c) $a+b+c$ (d) None of these

Solution: (a) $\theta=\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$

Let $s^2=\frac{a+b+c}{abc}$ $\quad\therefore\ \theta=\tan^{-1}\sqrt{a^2s^2}+\tan^{-1}\sqrt{b^2s^2}+\tan^{-1}\sqrt{c^2s^2}$

$\Rightarrow\ \theta=\tan^{-1}(as)+\tan^{-1}(bs)+\tan^{-1}(cs)\ \Rightarrow\ \theta=\tan^{-1}\left[\frac{as+bs+cs-abcs^3}{1-abs^2-bcs^2-cas^2}\right]$

$\Rightarrow\ \tan\theta=s\left[\frac{(a+b+c)-abcs^2}{1-(ab+bc+ca)s^2}\right]=0$ $\quad[\because abcs^2=(a+b+c)]$

Trick : Since it is an identity so it will be true for any value of *a*,*b*,c. Let $a=b=c=1$ then

$\theta = \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3} = \pi, \quad \tan\theta = 0$.

Example: 27 All possible values of p and q for which $\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$ holds, is

(a) $p = -1, q = \frac{1}{2}$ (b) $q > 1, p = \frac{1}{2}$ (c) $0 \le p \le 1, q = \frac{1}{2}$ (d) None of these

Solution: (c) $\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} = \frac{3\pi}{4} - \cos^{-1}\sqrt{1-q} \Rightarrow \cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) - \cos^{-1}\sqrt{1-q}$

$\Rightarrow \sqrt{p}\sqrt{1-p} - \sqrt{1-p}\sqrt{p} = -\left[\frac{1}{\sqrt{2}}.\sqrt{1-q} - \frac{1}{\sqrt{2}}.\sqrt{q}\right] \Rightarrow 0 = \sqrt{1-q} - \sqrt{q} \Rightarrow q = \frac{1}{2}$.

5.3 Inverse Trigonometric Ratios of Multiple Angles

(1) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$, If $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(2) $2\sin^{-1}x = \pi - \sin^{-1}(2x\sqrt{1-x^2})$, If $\frac{1}{\sqrt{2}} \le x \le 1$

(3) $2\sin^{-1}x = -\pi - \sin^{-1}(2x\sqrt{1-x^2})$, If $-1 \le x \le \frac{-1}{\sqrt{2}}$

(4) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, If $\frac{-1}{2} \le x \le \frac{1}{2}$

(5) $3\sin^{-1}x = \pi - \sin^{-1}(3x - 4x^3)$, If $\frac{1}{2} < x \le 1$

(6) $3\sin^{-1}x = -\pi - \sin^{-1}(3x - 4x^3)$, If $-1 \le x < -\frac{1}{2}$

(7) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$, If $0 \le x \le 1$

(8) $2\cos^{-1}x = 2\pi - \cos^{-1}(2x^2 - 1)$, if $-1 \le x \le 0$

(9) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$ If $\frac{1}{2} \le x \le 1$

(10) $3\cos^{-1}x = 2\pi - \cos^{-1}(4x^3 - 3x)$, If $-\frac{1}{2} \le x \le \frac{1}{2}$

(11) $3\cos^{-1}x = 2\pi + \cos^{-1}(4x^3 - 3x)$, If $-1 \le x \le -\frac{1}{2}$

(12) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, if $-1 < x \le 1$

(13) $2\tan^{-1}x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, If $x > 1$

(14) $2\tan^{-1}x = -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, If $x < -1$

(15) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, If $-1 \le x \le 1$

(16) $2\tan^{-1}x = \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, If $x > 1$

(17) $2\tan^{-1}x = -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, If $x < -1$

(18) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, If $0 \le x < \infty$

(19) $2\tan^{-1}x = -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, If $-\infty < x \le 0$

(20) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$, If $\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

(21) $3\tan^{-1}x = \pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, If $x > \frac{1}{\sqrt{3}}$ (22)

$3\tan^{-1}x = -\pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, If $x < -\frac{1}{\sqrt{3}}$

(23) $\tan^{-1}\left[\frac{x}{\sqrt{a^2 - x^2}}\right] = \sin^{-1}\frac{x}{a}$ (24)

$\tan^{-1}\left[\frac{3a^2x - x^3}{a(a^2 - 3x^2)}\right] = 3\tan^{-1}\frac{x}{a}$

(25) $\tan^{-1}\left[\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right] = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$ (26)

$\tan^{-1}\sqrt{\frac{1-x}{1+x}} = \frac{1}{2}\cos^{-1}x$

Example: 28 $2\tan^{-1}(\cos x) = \tan^{-1}(\text{cosec}^2 x)$, then $x =$ **[UPSEAT 2002]**

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

Solution: (d) $2\tan^{-1}(\cos x) = \tan^{-1}(\text{cos ec}^2 x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1 - \cos^2 x}\right) = \tan^{-1}\left(\frac{1}{\sin^2 x}\right) \Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 2\cos x = 1 \Rightarrow x = \frac{\pi}{3}$.

Example: 29 The solution set of the equation $\sin^{-1}x = 2\tan^{-1}x$ is **[AMU 2002]**

(a) {1, 2} (b) $\{-1, 2\}$ (c) $\{-1, 1, 0\}$ (d) $\{1, \frac{1}{2}, 0\}$

Soltuion: (c) $\sin^{-1}x = 2\tan^{-1}x \Rightarrow \sin^{-1}x = \sin^{-1}\frac{2x}{1+x^2} \Rightarrow \frac{2x}{1+x^2} = x \Rightarrow x^3 - x = 0 \Rightarrow x(x+1)(x-1) = 0 \Rightarrow x = \{-1, 1, 0\}$.

Exmaple: 30 $\sin\left\{\tan^{-1}\left(\frac{1-x^2}{2x}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$ is equal to **[Kurukshetra CEE 2001]**

(a) 0 (b) 1 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

Solution: (b) $\sin\left\{\left(\frac{\pi}{2} - 2\tan^{-1}x\right) + 2\tan^{-1}x\right\} = \sin\frac{\pi}{2} = 1$.

Example: 31 If $\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}\frac{2x}{1-x^2}$, then $x =$ **[EAMCET 1989]**

(a) a (b) b (c) $\frac{a+b}{1-ab}$ (d) $\frac{a-b}{1+ab}$

Solution: (d) Put $a = \tan\theta, b = \tan\phi$ and $x = \tan\psi$, then reduced form is

$\sin^{-1}(\sin 2\theta) - \cos^{-1}(\cos 2\phi) = \tan^{-1}(\tan 2\psi) \Rightarrow 2\theta - 2\phi = 2\psi \Rightarrow \theta - \phi = \psi$

Taking tan on both sides, we get $\tan(\theta - \phi) = \tan\psi \Rightarrow \frac{\tan\theta - \tan\phi}{1 + \tan\theta.\tan\phi} = \tan\psi$

Substituting these values, we get $\frac{a-b}{1+ab}=x$

Example: 32 $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]=$ **[IIT 1984]**

(a) $\frac{17}{7}$ (b) $-\frac{17}{7}$ (c) $\frac{7}{17}$ (d) $-\frac{7}{17}$

Solution: (d) $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]=\tan\left[\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}-\tan^{-1}(1)\right]=\tan\left[\tan^{-1}\frac{5}{12}-\tan^{-1}(1)\right]=\tan.\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)=\frac{-7}{17}$.

Example: 33 $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=$ **[Roorkee 1981]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) None of these

Solution: (c) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=2\tan^{-1}\left[\frac{\frac{2}{5}}{1-\frac{1}{25}}\right]-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$=2\tan^{-1}\left(\frac{5}{12}\right)-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\tan^{-1}\left[\frac{\frac{5}{6}}{1-\frac{25}{144}}\right]-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$=\tan^{-1}\left(\frac{120}{119}\right)-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\tan^{-1}\frac{120}{119}+\tan^{-1}\left[\frac{\frac{1}{99}-\frac{1}{70}}{1+\frac{1}{99}.\frac{1}{70}}\right]=\tan^{-1}\frac{120}{119}+\tan^{-1}\left(-\frac{29}{6931}\right)$

$=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{29}{6931}=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}=\tan^{-1}\left[\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right]=\tan^{-1}(1)=\frac{\pi}{4}$.

Example: 34 The value of $\sin\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)+\cos(\tan^{-1}2\sqrt{2})=$ **[AMU 1999]**

(a) $\frac{16}{15}$ (b) $\frac{14}{15}$ (c) $\frac{12}{15}$ (d) $\frac{11}{15}$

Solution: (b) $\sin\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]+\cos[\tan^{-1}(2\sqrt{2})]=\sin\left[\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}\right]+\cos[\tan^{-1}(2\sqrt{2})]$

$=\sin\left[\tan^{-1}\frac{3}{4}\right]+\cos[\tan^{-1}2\sqrt{2}]=\sin\left[\sin^{-1}\frac{3}{5}\right]+\cos\left[\cos^{-1}\frac{1}{3}\right]=\frac{3}{5}+\frac{1}{3}=\frac{14}{15}$.

Example: 35 $\tan\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right]+\tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right]$ equal to **[MP PET 1999]**

(a) $\frac{2a}{b}$ (b) $\frac{2b}{a}$ (c) $\frac{a}{b}$ (d) $\frac{b}{a}$

Solution: (b) Let $\cos^{-1}\frac{a}{b}=\theta\Rightarrow\cos\theta=\frac{a}{b}$

$\tan\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right]+\tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right]=\frac{1+t}{1-t}+\frac{1-t}{1+t}$, where $t=\tan\frac{\theta}{2}$ $=2\frac{(1+t^2)}{1-t^2}=\frac{2}{\cos\theta}=\frac{2b}{a}$.

Example: 36 $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$ is equal to

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

Solution: (d) Since, $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$

$$\therefore\quad 4\tan^{-1}\frac{1}{5}=2\left[2\tan^{-1}\frac{1}{5}\right]=2\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}=2\tan^{-1}\frac{10}{24}=\tan^{-1}\frac{\frac{20}{24}}{1-\frac{100}{576}}=\tan^{-1}\frac{120}{119}$$

$$\text{So, } 4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}=\tan^{-1}\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}=\tan^{-1}\frac{(120\times 239)-119}{(119\times 239)+120}=\tan^{-1}1=\frac{\pi}{4}.$$

Example: 37 The formula $\cos^{-1}\frac{1-x^2}{1+x^2}=2\tan^{-1}x$ holds only for

(a) $x\in R$ (b) $|x|\le 1$ (c) $x\in(-1,1]$ (d) $x\in[1,+\infty)$

Solution: (d) If $x=-1$, LHS = $\frac{\pi}{2}$, RHS = $2\times\left(-\frac{\pi}{2}\right)$. So, the fomula does not hold.

If $x<-1$, the angle on the LHS is in the second quadrant while the angle on the RHS is $2\times$ (angle in the fourth quadrant), which cannot be equal.

If $x>1$, the angle on the LHS is in the second quadrant while the angle on the RHS is $2\times$ (angle in the first quadrant) and these two may be equal.

If $-1<x<0$, the angle on the LHS is positive and that on the RHS is negative and the two cannot be equal.

Example: 38 $2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$ is independent of x, then

(a) $x\in[1,+\infty)$ (b) $x\in[-1,1]$ (c) $x\in(-\infty,-1]$ (d) None of these

Solution: (a) Let $x=\tan\theta$. Then $\sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}(\sin 2\theta)$

$$\therefore\quad 2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=2\theta+\sin^{-1}(\sin 2\theta)$$

If $-\frac{\pi}{2}\le 2\theta\le\frac{\pi}{2}$, $2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$ = $2\theta+2\theta=4\tan^{-1}x\ne$ independent of x.

If $-\frac{\pi}{2}\le\pi-2\theta\le\frac{\pi}{2}$, $2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$ = $2\theta+\sin^{-1}[\sin(\pi-2\theta)]=2\theta+\pi-2\theta=$ π = independent of x.

$\therefore\ \theta\notin\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$ but $\theta\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$ and from the principal value of $\tan^{-1}x$.

$\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Hence, $\theta\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

$$\therefore\quad \theta\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)\Rightarrow\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=\pi.$$

Also at $\theta=\frac{\pi}{4}$, $2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=2.\frac{\pi}{4}+\sin^{-1}1=\frac{\pi}{2}+\frac{\pi}{2}=\pi$.

$\therefore$ The given function = π = constant if $\theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)$. *i.e.,* $x\in[1,+\infty)$.

Example: 39 The number of positive integral solutions of the equation $\tan^{-1} x + \cos^{-1} \frac{y}{\sqrt{1+y^2}} = \sin^{-1} \frac{3}{\sqrt{10}}$ or $\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3$ is

(a) One (b) Two (c) Zero (d) None of these

Solution: (b) $\tan^{-1} x + \tan^{-1} \frac{1}{y} = \tan^{-1} 3$ or $\tan^{-1} \frac{1}{y} = \tan^{-1} 3 - \tan^{-1} x$ or $\tan^{-1} \frac{1}{y} = \tan^{-1} \frac{3-x}{1+3x} \Rightarrow y = \frac{1+3x}{3-x}$

As x, y are positive integers, $x = 1$, 2 and corresponding $y = 2$, 7

$\therefore$ Solutions are $(x, y) = (1, 2), (2, 7)$.

Example: 40 α, β and γ are three angles given by $\alpha = 2\tan^{-1}(\sqrt{2}-1), \beta = 3\sin^{-1} \frac{1}{\sqrt{2}} + \sin^{-1}\left(-\frac{1}{2}\right)$ and $\gamma = \cos^{-1}\left(\frac{1}{3}\right)$. Then

(a) $\alpha > \beta$ (b) $\beta > \gamma$ (c) $\alpha < \gamma$ (d) None of these

Solution: (b,c) $\alpha = 2\tan^{-1}(\sqrt{2}-1) = 2\tan^{-1} \tan \frac{\pi}{8} = 2 \times \frac{\pi}{8} = \frac{\pi}{4} = \cos^{-1} \frac{1}{\sqrt{2}}$

$\beta = 3.\frac{\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}$

$\therefore$ $\beta > \alpha$. Also, $\frac{1}{3} < \frac{1}{\sqrt{2}} \Rightarrow \cos^{-1} \frac{1}{3} > \cos^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$

So, $\gamma > \alpha$

Again $\cos^{-1}\left(\frac{1}{3}\right)$ belongs to the first quadrant and β is in the second quadrant.

$\therefore \beta > \gamma$.

Example: 41 $\frac{a^3}{2} cosec^2\left(\frac{1}{2}\tan^{-1}\frac{a}{b}\right) + \frac{b^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{b}{a}\right)$ is equal to

(a) $(a-b)(a^2+b^2)$ (b) $(a+b)(a^2-b^2)$ (c) $(a+b)(a^2+b^2)$ (d) None of these

Soltuion: (c) Let $\tan^{-1}\frac{a}{b} = \theta, \tan^{-1}\frac{b}{a} = \phi \Rightarrow \therefore \tan\theta = \frac{a}{b}, \tan\phi = \frac{b}{a}$

$\frac{a^3}{2} \text{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{a}{b}\right) + \frac{b^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{b}{a}\right)$

$$= \frac{a^3}{2\sin^2\left(\frac{\theta}{2}\right)} + \frac{b^3}{2\cos^2\left(\frac{\phi}{2}\right)} = \frac{a^3}{1-\cos\theta} + \frac{b^3}{1+\cos\phi} = \frac{a^3}{1-\frac{b}{\sqrt{a^2+b^2}}} + \frac{b^3}{1+\frac{a}{\sqrt{a^2+b^2}}}$$

$$= \sqrt{a^2+b^2}\left[\frac{a^3[\sqrt{a^2+b^2}+b]}{(a^2+b^2)-b^2} + \frac{b^3[\sqrt{a^2+b^2}-a]}{(a^2+b^2)-a^2}\right] \quad \text{(rationalized)}$$

$$= \sqrt{a^2+b^2}[a\{\sqrt{a^2+b^2}+b\} + b\{\sqrt{a^2+b^2}-a\}] = \sqrt{a^2+b^2}[\sqrt{a^2+b^2}(a+b)] = (a^2+b^2)(a+b).$$

Assignment

Properties of Inverse Trigonometrical Function

Basic Level

1. The domain of $\sin^{-1} x$ is **[Roorkee Screening 1993]**

(a) $(-\pi, \pi)$ (b) [– 1, 1] (c) $(0, 2\pi)$ (d) $(-\infty, \infty)$

2. The range of $\tan^{-1} x$ is **[DCE 2002]**

(a) $\left(\pi, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (c) $(-\pi, \pi)$ (d) $(0, \pi)$

3. $\sin^{-1} x + \cos^{-1} x$ is equal to **[Pb. CET 1997; DCE 2002]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) – 1 (d) 1

4. $\sin\left\{\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\right\} =$ **[EAMCET 1985]**

(a) 0 (b) – 1 (c) 2 (d) 1

5. The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\cos\frac{5\pi}{3}\right)$ is **[UPSEAT 2003]**

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{3}$ (c) $\frac{10\pi}{3}$ (d) 0

6. $\cos\left[\cos^{-1}\left(\frac{-1}{7}\right) + \sin^{-1}\left(\frac{-1}{7}\right)\right] =$ **[EAMCET 2003]**

(a) $-\frac{1}{3}$ (b) 0 (c) $\frac{1}{3}$ (d) $\frac{4}{9}$

7. The value of $\tan^{-1} x + 2\cot^{-1} x$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) 2π

8. If $\tan^{-1} x + 2\cot^{-1} x = \frac{2\pi}{3}$, then $x =$ **[Karnataka CET 1999]**

(a) $\sqrt{2}$ (b) 3 (c) $\sqrt{3}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

9. If $4\sin^{-1} x + \cos^{-1} x = \pi$, then x is equal to **[UPSEAT 2001]**

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$

10. $\cos\left[2\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5}\right] =$ **[IIT 1981]**

(a) $\frac{2\sqrt{6}}{5}$ (b) $-\frac{2\sqrt{6}}{5}$ (c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

11. The value of $\sin^{-1}\left(\cos\frac{33\pi}{5}\right)$ is

(a) $\frac{3\pi}{5}$ (b) $\frac{7\pi}{5}$ (c) $\frac{\pi}{10}$ (d) $-\frac{\pi}{10}$

12. If $\sec^{-1}\left(\frac{1}{x}\right)+2\sin^{-1}(1)=\pi$, then x equals **[AMU 1988]**

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{\pi}{2}$ (d) None of these

13. The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)$ is **[MP PET 2003]**

(a) 45^o (b) 90^o (c) 15^o (d) 30^o

14. The value of $\cos(\tan^{-1}(\tan 2))$ is

(a) $\frac{1}{\sqrt{5}}$ (b) $-\frac{1}{\sqrt{5}}$ (c) $\cos 2$ (d) $-\cos 2$

15. The value of x which satisfies the equation $\tan^{-1}x=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ is

(a) 3 (b) – 3 (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

16. If $\sec^{-1}x=\text{cosec}^{-1}y$, then $\cos^{-1}\frac{1}{x}+\cos^{-1}\frac{1}{y}=$ **[Orissa JEE 2002]**

(a) π (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

17. If $\cos^{-1}\left(\frac{1}{x}\right)=\theta$, then $\tan\theta=$ **[MNR 1978; MP PET 1989]**

(a) $\frac{1}{\sqrt{x^2-1}}$ (b) $\sqrt{x^2+1}$ (c) $\sqrt{1-x^2}$ (d) $\sqrt{x^2-1}$

18. If $\sin^{-1}x=\frac{\pi}{5}$ for some $x\in(-1,1)$, then the value of $\cos^{-1}x$ is **[DCE 1997; Karnataka CET 1996; IIT 1992]**

(a) $\frac{3\pi}{10}$ (b) $\frac{5\pi}{10}$ (c) $\frac{7\pi}{10}$ (d) $\frac{9\pi}{10}$

19. $\sec(\text{cosec}^{-1}x)$ is equal to **[Kurukshetra CEE 2001]**

(a) $\text{cosec}(\sec^{-1}x)$ (b) $\cot x$ (c) π (d) None of these

20. $\tan(\cos^{-1}x)$ is equal to **[IIT 1993]**

(a) $\frac{\sqrt{1-x^2}}{x}$ (b) $\frac{x}{1+x^2}$ (c) $\frac{\sqrt{1+x^2}}{x}$ (d) $\sqrt{1-x^2}$

21. $\sin(\cot^{-1}x)=$ **[MNR 1987; MP PET 2001; DCE 2002]**

(a) $\sqrt{1+x^2}$ (b) x (c) $(1+x^2)^{-3/2}$ (d) $(1+x^2)^{-\frac{1}{2}}$

22. $\cos(\tan^{-1} x) =$ **[MP PET 1988; MNR 1981]**

(a) $\sqrt{1+x^2}$ (b) $\frac{1}{\sqrt{1+x^2}}$ (c) $1+x^2$ (d) None of these

23. $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right] =$ **[Karnataka CET 1994]**

(a) $\frac{25}{24}$ (b) $\frac{25}{7}$ (c) $\frac{24}{25}$ (d) None of these

24. The value of $\sin \cot^{-1} \tan \cos^{-1} x$ is equal to **[Bihar CEE 1974]**

(a) x (b) $\frac{\pi}{2}$ (c) 1 (d) None of these

25. $\left[\sin\left(\tan^{-1}\frac{3}{4}\right)\right]^2 =$ **[EAMCET 1983]**

(a) $\frac{3}{5}$ (b) $\frac{5}{3}$ (c) $\frac{9}{25}$ (d) $\frac{25}{9}$

26. $\sec^2(\tan^{-1} 2) + \text{cosec}^2(\cot^{-1} 3) =$ **[EAMCET 2001]**

(a) 5 (b) 13 (c) 15 (d) 6

Advance Level

27. If $\cos^{-1} x > \sin^{-1} x$, then

(a) $x < 0$ (b) $-1 < x < 0$ (c) $0 \le x < \frac{1}{\sqrt{2}}$ (d) $-1 \le x < \frac{1}{\sqrt{2}}$

28. If $(\cos^{-1} x)^2 - (\sin^{-1} x)^2 > 0$, then

(a) $x < \frac{1}{2}$ (b) $-1 < x < \sqrt{2}$ (c) $0 \le x < \frac{1}{\sqrt{2}}$ (d) $-1 \le x < \frac{1}{\sqrt{2}}$

29. The greatest and the least values of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$ are

(a) $\frac{-\pi}{2}, \frac{\pi}{2}$ (b) $\frac{-\pi^3}{8}, \frac{\pi^3}{8}$ (c) $\frac{\pi^3}{32}, \frac{7\pi^3}{8}$ (d) None of these

30. If x satisfies the equation $t^2 - t - 2 > 0$, then there exists a value for

(a) $\sin^{-1} x$ (b) $\cos^{-1} x$ (c) $\sec^{-1} x$ (d) None of these

31. If $f(x) = \sec^{-1} x + \tan^{-1} x$, then $f(x)$ is real for

(a) $x \in [-1, 1]$ (b) $x \in R$ (c) $x \in (-\infty, 1] \cup [1, +\infty)$ (d) None of these

32. If $\sum_{r=1}^{2n} \sin^{-1} x_r = n\pi,$ then $\sum_{r=1}^{2n} x_r =$

(a) n (b) $2n$ (c) $\frac{n(n+1)}{2}$ (d) None of these

33. $\frac{-2\pi}{5}$ is the principal value of

(a) $\cos^{-1}(\cos \frac{7\pi}{5})$ (b) $\sin^{-1}(\sin \frac{7\pi}{5})$ (c) $\sec^{-1}(\sec \frac{7\pi}{5})$ (d) None of these

34. The number of real solutions of (x, y) ; where $|y| = \sin x, y = \cos^{-1}(\cos x), -2\pi \le x \le 2\pi$ is

(a) 2 (b) 1 (c) 3 (d) 4

35. The set of values of k for which $x^2 - kx + \sin^{-1}(\sin 4) > 0$ for all real x is

(a) ϕ (b) $(-2, 2)$ (c) R (d) None of these

36. $\cos^{-1}\left\{\frac{1}{2}x^2 + \sqrt{1-x^2}.\sqrt{1-\frac{x^2}{4}}\right\} = \cos^{-1}\frac{x}{2} - \cos^{-1}x$ holds for

(a) $|x| \le 1$ (b) $x \in R$ (c) $0 \le x \le 1$ (d) $-1 \le x \le 0$

Sum and Difference of Inverse Trigonometrical

Basic Level

37. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi,$ then $xy + yz + zx =$ **[Karnataka CET 2003]**

(a) 0 (b) 1 (c) 3 (d) – 3

38. The value of $\tan\left(\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3}\right)$ is **[AMU 2001]**

(a) $\frac{5}{6}$ (b) $\frac{7}{6}$ (c) $\frac{1}{6}$ (d) $\frac{1}{7}$

39. $\tan^{-1}\left(\frac{1}{11}\right) + \tan^{-1}\left(\frac{2}{12}\right) =$ **[DCE 1999]**

(a) $\tan^{-1}\left(\frac{33}{132}\right)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$ (c) $\tan^{-1}\left(\frac{132}{33}\right)$ (d) None of these

40. If $\sin^{-1}a + \sin^{-1}b + \sin^{-1}c = \pi,$ then the value of $a\sqrt{(1-a^2)} + b\sqrt{(1-b^2)} + c\sqrt{(1-c^2)}$ will be **[UPSEAT 1999]**

(a) $2abc$ (b) abc (c) $\frac{1}{2}abc$ (d) $\frac{1}{3}abc$

41. If $\tan^{-1}x - \tan^{-1}y = \tan^{-1}A,$ then $A =$ **[MP PET 1988]**

(a) $x - y$ (b) $x + y$ (c) $\frac{x-y}{1+xy}$ (d) $\frac{x+y}{1-xy}$

42. If $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$ then $x =$ **[Roorkee 1978, 1980; MNR 1986; Karnataka CET 2002]**

(a) –1 (b) $\frac{1}{6}$ (c) $-1, \frac{1}{6}$ (d) None of these

43. If $\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{4}{5} = \cos^{-1}x$, then $x =$ **[AMU 1978]**

(a) 0 (b) 1 (c) –1 (d) 2

44. If $\cot^{-1}\alpha + \cot^{-1}\beta = \cot^{-1}x$, then $x =$ **[MP PET 1992]**

(a) $\alpha + \beta$ (b) $\alpha - \beta$ (c) $\frac{1+\alpha\beta}{\alpha+\beta}$ (d) $\frac{\alpha\beta - 1}{\alpha+\beta}$

45. $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} =$ **[AMU 1976, 1977]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) None of these

46. $\cos\left[\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right] =$ **[MP PET 1991; MNR 1990]**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{\pi}{4}$

47. $\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y} =$ (where $x > y > 0$) **[EAMCET 1992]**

(a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) None of these

48. $\tan\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right] =$ **[IIT 1983; EAMCET 1988; MP PET 1990; MNR 1992]**

(a) $\frac{6}{17}$ (b) $\frac{17}{6}$ (c) $\frac{7}{16}$ (d) $\frac{16}{7}$

49. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) =$ **[EAMCET 1994]**

(a) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (c) $\tan^{-1}\left(\frac{1}{2}\right)$ (d) Both (a) and (c)

50. $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3$ is equal to **[Karnataka CET 1995; MP PET 1993]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

51. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then $x =$ **[UPSEAT 1994]**

(a) 1 (b) 0 (c) $\frac{4}{5}$ (d) $\frac{1}{5}$

52. A solution of the equation $\tan^{-1}(1+x) + \tan^{-1}(1-x) = \frac{\pi}{2}$ is **[Karnataka CET 1993]**

(a) $x = 1$ (b) $x = -1$ (c) $x = 0$ (d) $x = \pi$

53. If $\tan^{-1}\frac{x-1}{x+1}+\tan^{-1}\frac{2x-1}{2x+1}=\tan^{-1}\frac{23}{36}$, then x = **[ISM Dhanbad 1973]**

(a) $\frac{3}{4},\frac{-3}{8}$ (b) $\frac{3}{4},\frac{3}{8}$ (c) $\frac{4}{3},\frac{3}{8}$ (d) None of these

54. If $\sin^{-1}\frac{x}{5}+cosec^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then $x=$ **[EAMCET 1983]**

(a) 4 (b) 5 (c) 1 (d 3

55. $\sin^{-1}\left(\frac{3}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)=$ **[Karnataka CET 1994]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\cos^{-1}\left(\frac{4}{5}\right)$ (d) π

56. If $\tan^{-1}2, \tan^{-1}3$ are two angles of a triangle, then the third angle is

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{4}$ (d) None of these

57. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ then

(a) $x^2+y^2+z^2+xyz=0$ (b) $x^2+y^2+z^2+2xyz=0$ (c) $x^2+y^2+z^2+xyz=1$ (d) $x^2+y^2+z^2+2xyz=1$

58. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then **[Karnataka CET 1996]**

(a) $x+y+z-xyz=0$ (b) $x+y+z+xyz=0$ (c) $xy+yz+zx+1=0$ (d) $xy+yz+zx-1=0$

59. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=$ **[MP PET 1991]**

(a) 0 (b) 1 (c) $\frac{1}{xyz}$ (d) xyz

Advance Level

60. If we consider only the principal values of the inverse trigonometric functions, then the value of $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{(17)}}\right)$ is

[IIT 1994]

(a) $\sqrt{\frac{29}{3}}$ (b) $\frac{29}{3}$ (c) $\sqrt{\frac{3}{29}}$ (d) $\frac{3}{29}$

61. The sum of first 10 terms of the series $\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\cot^{-1}21+.......$ is **[Karnataka CET 1996]**

(a) $\tan^{-1}\left(\frac{5}{6}\right)$ (b) $\tan^{-1}(100)$ (c) $\tan^{-1}\left(\frac{6}{5}\right)$ (d) $\tan^{-1}\left(\frac{1}{100}\right)$

62. Sum of infinite terms of the series $\cot^{-1}\left[1^2+\frac{3}{4}\right]+\cot^{-1}\left[2^2+\frac{3}{4}\right]+\cot^{-1}\left[3^2+\frac{3}{4}\right]+$ ....is

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}2$ (c) $\tan^{-1}3$ (d) None of these

63. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{18}+........+\tan^{-1}\left(\frac{1}{n^2+n+1}\right)+.......$ to ∞ is equal **[Karnataka CET 2000]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{2\pi}{3}$ (d) 0

64. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then $x^4+y^4+z^4+4x^2y^2z^2=K(x^2y^2+y^2z^2+z^2x^2)$, where K =

(a) 1 (b) 2 (c) 4 (d) None of these

65. The sum of the infinite series $\sin^{-1}\frac{1}{\sqrt{2}}+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)+.........\ ....+\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

66. If sum of the infinite series $\cot^{-1}(2.1^2)+\cot^{-1}(2.2^2)+\cot^{-1}(2.2^3)+\cot(2.2^4)+.....$ is equal to

(a) $\frac{\pi}{5}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

67. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then the value of $x^{100}+y^{100}+z^{100}-\frac{9}{x^{101}+y^{101}+z^{101}}$ is equal to

(a) 0 (b) 3 (c) – 3 (d) 9

68. If x_1, x_2, x_3, x_4 are roots of the equation $x^4-x^3\sin 2\beta+x^2\cos 2\beta-x\cos\beta-\sin\beta=0$,

then $\tan^{-1}x_1+\tan^{-1}x_2+\tan^{-1}x_3+\tan^{-1}x_4=$

(a) β (b) $\frac{\pi}{2}-\beta$ (c) $\pi-\beta$ (d) $-\beta$

69. If $a_1, a_2, a_3, ..., a_n$ is an A.P. with common difference *d*, then $\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+.........+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]=$

(a) $\frac{(n-1)d}{a_1+a_n}$ (b) $\frac{(n-1)d}{1+a_1a_n}$ (c) $\frac{nd}{1+a_1a_n}$ (d) $\frac{a_n-a_1}{a_n+a_1}$

Inverse Trigonometric Ratios of Multiple Angles

Basic Level

70. $3\tan^{-1}a$ is equal to **[MP PET 1993]**

(a) $\tan^{-1}\frac{3a+a^3}{1+3a^2}$ (b) $\tan^{-1}\frac{3a-a^3}{1+3a^2}$ (c) $\tan^{-1}\frac{3a+a^3}{1-3a^2}$ (d) $\tan^{-1}\frac{3a-a^3}{1-3a^2}$

71. If $A=\tan^{-1}x$, then $\sin 2A=$ **[MNR 1988; UPSEAT 2000]**

(a) $\frac{2x}{\sqrt{1-x^2}}$ (b) $\frac{2x}{1-x^2}$ (c) $\frac{2x}{1+x^2}$ (d) None of these

72. $\sin(2\sin^{-1}0.8)=$ **[MNR 1980]**

(a) 0.96 (b) 0.48 (c) 0.64 (d) None of these

73. If $\cos(2\sin^{-1}x)=\frac{1}{9}$, then $x=$ **[Roorkee 1975]**

(a) Only $\frac{2}{3}$ (b) Only $-\frac{2}{3}$ (c) $\frac{2}{3},-\frac{2}{3}$ (d) Neither $\frac{2}{3}$ nor $-\frac{2}{3}$

74. $\cos^{-1}\left(\frac{15}{17}\right)+2\tan^{-1}\left(\frac{1}{5}\right)=$ **[EAMCET 1981]**

(a) $\frac{\pi}{2}$ (b) $\cos^{-1}\left(\frac{171}{221}\right)$ (c) $\frac{\pi}{4}$ (d) None of these

75. $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=$ **[EAMCET 1983]**

(a) $\tan^{-1}\left(\frac{49}{29}\right)$ (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{4}$

76. $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=$ **[Dhanbad Engg. 1971]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) None of these

77. If $\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x$, then $x=$ **[MNR 1984]**

(a) $\frac{a-b}{1+ab}$ (b) $\frac{b}{1+ab}$ (c) $\frac{b}{1-ab}$ (d) $\frac{a+b}{1-ab}$

78. If $3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right)-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{3}$, then x equals **[Pb. CET 2001; AMU 1992]**

(a) 1 (b) 2 (c) 3 (d) $\sqrt{3}$

79. $\cot^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]=$ **[UPSEAT 1986]**

(a) $\pi-x$ (b) $2\pi-x$ (c) $\frac{\pi}{2}$ (d) $\pi-\frac{x}{2}$

80. $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)=$ **[Karnataka CET 2003]**

(a) $\frac{1}{\sqrt{10}}$ (b) $-\frac{1}{\sqrt{10}}$ (c) $\frac{1}{10}$ (d) $-\frac{1}{10}$

Advance Level

81. $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right]=$ **[ISM Dhanbad 1976]**

(a) $\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$ (b) $\cos^{-1}\left(\frac{a+b\cos\theta}{a\cos\theta+b}\right)$ (c) $\cos^{-1}\left(\frac{a\cos\theta}{a+b\cos\theta}\right)$ (d) $\cos^{-1}\left(\frac{b\cos\theta}{a\cos\theta+b}\right)$

82. If $\cot^{-1}[(\cos\alpha)^{1/2}]-\tan^{-1}[(\cos\alpha)^{1/2}]=x$. Then $\sin x=$ **[AIEEE 2002]**

(a) $\tan^2\left(\frac{\alpha}{2}\right)$ (b) $\cot^2\left(\frac{\alpha}{2}\right)$ (c) $\tan\alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$

83. The value of $\sin^{-1}\left\{\left(\sin\frac{\pi}{3}\right)\frac{x}{\sqrt{x^2+k^2-kx}}\right\}-\cos^{-1}\left\{\cos\frac{\pi}{6}\frac{x}{\sqrt{x^2+k^2-kx}}\right\}$, $\left(\text{where } \frac{k}{2}<x<2k, k>0\right)$ is

(a) $\tan^{-1}\left(\frac{2x^2+xk-k^2}{x^2-2xk+k^2}\right)$ (b) $\tan^{-1}\left(\frac{x^2+2xk-k^2}{x^2-2xk+k^2}\right)$ (c) $\tan^{-1}\left(\frac{x^2+2xk-2k^2}{2x^2-2xk+2k^2}\right)$ (d) None of these

84. Solution of equation $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}]=0$ is **[UPSEAT 1998; Roorkee 1992]**

(a) $x=\pm1$ only (b) $x=1\pm\sqrt{2}$ only (c) $x=(-1\pm\sqrt{2})$ only (d) All of these

85. The greater of the two angles $A=2\tan^{-1}(2\sqrt{2}-1)$ and $B=3\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{3}{5}\right)$ is **[IIT 1992]**

(a) B (b) A (c) C (d) None of these

86. $\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]=$ **[Roorkee 1986]**

(a) $\frac{3-\sqrt{5}}{2}$ (b) $\frac{3+\sqrt{5}}{2}$ (c) $\frac{2}{3-\sqrt{5}}$ (d) $\frac{2}{3+\sqrt{5}}$

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Answer Sheet

Inverse Trigonometrical Functions ***Assignment (Basic & Advance Level)***

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
b	b	b	d	a	b	c	c	b	b	d	b	d	c	a	d	d	a	a	a
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
d	b	d	a	c	c	d	d	c	c	c	b	b	c	a	c	c	d	d	a
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
c	b	b	d	a	a	b	b	d	b	d	c	d	d	a	b	d	d	b	d
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
a	b	b	b	c	b	a	b	b	d	c	a	c	d	d	a	d	b	d	a
81	82	83	84	85	86														
a	a	c	d	b	d														