Chapter 7

Three Dimensional Co-ordinate Geometry

Contents

Pierre Fermat

Rene' Descartes (1596-1650 A.D.), the father of analytical geometry, essentially dealt with plane geometry only in 1637. The same is true of his coinventor Pierre Fermet (1601-1665 A.D.)

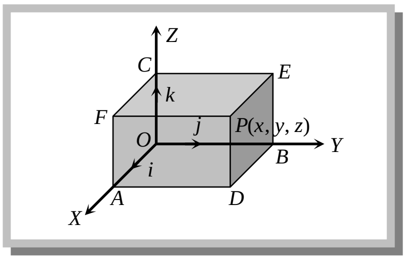
Descartes had the idea of co-ordinates in three dimensions but did not develop it.
J.Bernoulli (1667-1748 A.D.) in a letter of 1715 A.D. to Leibnitz introduced the three co-ordinate planes which we use today. It was Antoinne Parent (1666-1716 A.D.), who gave a systematic development of analytical solid geometry for the first time in a paper presented to the French Academy in 1700 A.D.
L.Euler (1707-1783 A.D.) took up systematically the three dimensional co-ordinate geometry.
It was not until the middle of the nineteenth century that geometry was extended to more than three dimensions, the well-known application of which is in the Space-Time Continuum of Einstein's Theory of Relativity.

Three Dimensional Co-ordinate Geometry

System of Co-ordinates

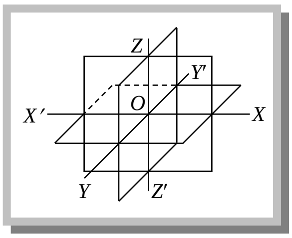
7.1 Co-ordinates of a Point in Space

(1) **Cartesian Co-ordinates :** Let O be a fixed point, known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis respectively, in such a way that they form a right-handed system.



The planes XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane respectively.

Let P be a point in space and distances of P from yz, zx and xy-planes be x, y, z respectively (with proper signs), then we say that co-ordinates of P are (x, y, z).



Also $OA = x$, $OB = y$, $OC = z$.

The three co-ordinate planes (XOY, YOZ and ZOX) divide space into eight parts and these parts are called octants.

Signs of co-ordinates of a point : The signs of the co-ordinates of a point in three dimension follow the convention that all distances measured along or parallel to OX, OY, OZ will be positive and distances moved along or parallel to OX', OY', OZ' will be negative.

The following table shows the signs of co-ordinates of points in various octants :

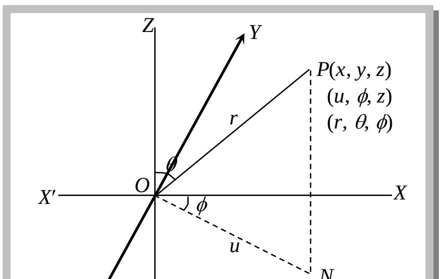
Octant co-ordinate	OXYZ	OX′YZ	OXY′Z	OX′Y′Z	OXYZ′	OX′YZ′	OXY′Z′	OX′Y′Z′
x	+	–	+	–	+	–	+	–
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

(2) **Other methods of defining the position of any point *P* in space :**

(i) **Cylindrical co-ordinates :** If the rectangular cartesian co-ordinates of P are (x, y, z), then those of N are $(x, y, 0)$ and we can easily have the following relations : $x = u\cos\phi$, $y = u\sin\phi$ and $z = z$.

Hence, $u^2 = x^2 + y^2$ and $\phi = \tan^{-1}(y/x)$.

Cylindrical co-ordinates of $P \equiv (u, \phi, z)$



(ii) **Spherical polar co-ordinates :** The measures of quantities r, θ, ϕ are known as spherical or three dimensional polar co-ordinates of the point P. If the rectangular cartesian co-ordinates of P are (x, y, z) then

$z = r\cos\theta$, $u = r\sin\theta$ $\therefore$ $x = u\cos\phi = r\sin\theta\cos\phi$, $y = u\sin\phi = r\sin\theta\sin\phi$ and $z = r\cos\theta$

Also $r^2 = x^2 + y^2 + z^2$ and $\tan\theta = \frac{u}{z} = \frac{\sqrt{x^2+y^2}}{z}$; $\tan\phi = \frac{y}{x}$

Note : ❑ The co-ordinates of a point on xy-plane is $(x, y, 0)$, on yz-plane is $(0, y, z)$ and on zx-plane is $(x, 0, z)$

❑ The co-ordinates of a point on x-axis is $(x, 0, 0)$, on y-axis is $(0, y, 0)$ and on z-axis is $(0, 0, z)$

❑ **Position vector of a point :** Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be unit vectors along OX, OY and OZ respectively. Then position vector of a point $P(x, y, z)$ is $\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

7.2 Distance Formula

(1) **Distance formula :** The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

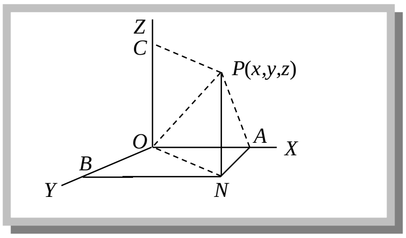
(2) **Distance from origin :** Let O be the origin and $P(x, y, z)$ be any point, then $OP = \sqrt{(x^2 + y^2 + z^2)}$.

(3) **Distance of a point from co-ordinate axes :** Let $P(x, y, z)$ be any point in the space. Let PA, PB and PC be the perpendiculars drawn from P to the axes OX, OY and OZ respectively.

Then, $PA = \sqrt{(y^2 + z^2)}$

$PB = \sqrt{(z^2 + x^2)}$

$PC = \sqrt{(x^2 + y^2)}$



Example: 1 The distance of the point (4, 3, 5) from the y-axis is **[MP PET 2003]**

(a) $\sqrt{34}$ (b) 5 (c) $\sqrt{41}$ (d) $\sqrt{15}$

Solution: (c) Distance $= \sqrt{x^2 + z^2} = \sqrt{16 + 25} = \sqrt{41}$

Example: 2 The points (5, –4, 2), (4, –3, 1), (7, –6, 4) and (8, –7, 5) are the vertices of **[Rajasthan PET 2002]**

(a) A rectangle (b) A square (c) A parallelogram (d) None of these

Solution: (c) Let the points be $A(5, -4, 2)$, $B(4, -3, 1)$, $C(7, -6, 4)$ and $D(8, -7, 5)$.

$AB = \sqrt{1+1+1} = \sqrt{3}$, $CD = \sqrt{1+1+1} = \sqrt{3}$, $BC = \sqrt{9+9+9} = 3\sqrt{3}$, $AD = \sqrt{9+9+9} = 3\sqrt{3}$

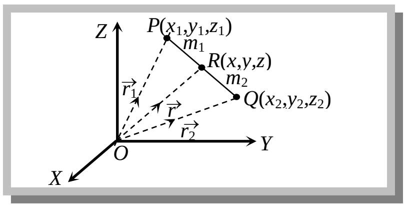
Length of diagonals $AC = \sqrt{4+4+4} = 2\sqrt{3}$, $BD = \sqrt{16+16+16} = 4\sqrt{3}$

i.e., $AC \neq BD$

Hence, A, B, C, D are vertices of a parallelogram

7.3 Section Formulas

(1) **Section formula for internal division :** Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Let R be a point on the line segment joining P and Q such that it divides the join of P and Q internally in the ratio $m_1 : m_2$. Then the co-ordinates of R are $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$.



(2) **Section formula for external division :** Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points, and let R be a point on PQ produced, dividing it externally in the ratio $m_1 : m_2$ $(m_1 \neq m_2)$. Then the co-ordinates of R are $\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right)$.

Note : ❑ **Co-ordinates of the midpoint :** When division point is the mid-point of PQ then ratio will be 1 : 1, hence co-ordinates of the mid point of PQ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

❑ **Co-ordinates of the general point :** The co-ordinates of any point lying on the line joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as $\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$, which divides PQ in the ratio k : 1. This is called general point on the line PQ.

Example: 3 If the x-co-ordinate of a point P on the join of Q (2, 2, 1) and R (5, 1, –2) is 4, then its z-co-ordinate is **[Rajasthan PET 2003]**

(a) 2 (b) 1 (c) –1 (d) –2

Solution: (c) Let the point P be $\left(\frac{5k+2}{k+1}, \frac{k+2}{k+1}, \frac{-2k+1}{k+1}\right)$. $\because$ Given that $\frac{5k+2}{k+1} = 4 \Rightarrow k = 2$ $\therefore$ z-co-ordinate of $P = \frac{-2(2)+1}{2+1} = -1$

7.4 Triangle

(1) **Co-ordinates of the centroid**

(i) If $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) are the vertices of a triangle, then co-ordinates of its centroid are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

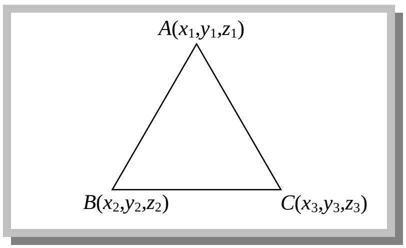
(ii) If (x_r, y_r, z_r); r = 1, 2, 3, 4, are vertices of a tetrahedron, then co-ordinates of its centroid are $\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$.

(iii) If G (α, β, γ) is the centroid of ΔABC, where A is (x_1, y_1, z_1), B is (x_2, y_2, z_2), then C is $(3\alpha - x_1 - x_2, 3\beta - y_1 - y_2, 3\gamma - z_1 - z_2)$.

(2) **Area of triangle** : Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of a triangle, then

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, area of ΔABC is given by the relation $\Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$.



Also, $\Delta = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}\right\|$

(3) **Condition of collinearity** : Points $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are collinear

If $\frac{x_1 - x_2}{x_2 - x_3} = \frac{y_1 - y_2}{y_2 - y_3} = \frac{z_1 - z_2}{z_2 - z_3}$

7.5 Volume of Tetrahedron

Volume of tetrahedron with vertices (x_r, y_r, z_r); $r = 1, 2, 3, 4$, is $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

Example: 4 If centroid of tetrahedron *OABC*, where *A*, *B*, *C* are given by (*a*, 2, 3), (1, *b*, 2) and (2, 1, *c*) respectively be (1, 2, –1), then distance of *P*(*a*, *b*, *c*) from origin is equal to

(a) $\sqrt{107}$ (b) $\sqrt{14}$ (c) $\sqrt{107/14}$ (d) None of these

Solution: (a) (1, 2, –1) is the centroid of the tetrahedron

$\therefore\ 1 = \frac{0 + a + 1 + 2}{4} \Rightarrow a = 1,\ 2 = \frac{0 + 2 + b + 1}{4} \Rightarrow b = 5,\ -1 = \frac{0 + 3 + 2 + c}{4} \Rightarrow c = -9.$

$\therefore$ (*a*, *b*, *c*) = (1, 5, –9). Its distance from origin $= \sqrt{1 + 25 + 81} = \sqrt{107}$

Example: 5 If vertices of triangle are $A(1, -1, 2)$, $B(2, 0, -1)$ and $C(0, 2, 1)$, then the area of triangle is **[Rajasthan PET 2000]**

(a) $\sqrt{6}$ (b) $2\sqrt{6}$ (c) $3\sqrt{6}$ (d) $4\sqrt{6}$

Solution: (b) $\Delta = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (2-1) & (0+1) & (-1-2) \\ (0-2) & (2-0) & (1+1) \end{vmatrix}\right\| = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -2 & 2 & 2 \end{vmatrix}\right\| = \frac{1}{2}|\mathbf{i}(8) - \mathbf{j}(-4) + \mathbf{k}(4)|$

$= |4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}| = \sqrt{16 + 4 + 4} = \sqrt{24} = 2\sqrt{6}$

Example: 6 The points (5, 2, 4), (6, –1, 2) and (8, –7, *k*) are collinear, if *k* is equal to **[Kurukshetra CEE 2000]**

(a) –2 (b) 2 (c) 3 (d) –1

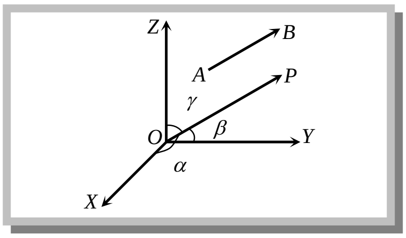
Solution: (a) If given points are collinear, then

$\frac{x_1 - x_2}{x_2 - x_3} = \frac{y_1 - y_2}{y_2 - y_3} = \frac{z_1 - z_2}{z_2 - z_3} \Rightarrow \frac{5-6}{6-8} = \frac{2+1}{-1+7} = \frac{4-2}{2-k} \Rightarrow \frac{-1}{-2} = \frac{3}{6} = \frac{2}{2-k} \Rightarrow \frac{1}{2} = \frac{2}{2-k} \Rightarrow k = -2$

7.6 Direction cosines and Direction ratio

(1) Direction cosines

(i) The cosines of the angle made by a line in anticlockwise direction with positive direction of co-ordinate axes are called the direction cosines of that line.



If α, β, γ be the angles which a given directed line makes with the positive direction of the *x*, *y*, *z* co-ordinate axes respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the given line and are generally denoted by *l*, *m*, *n* respectively.

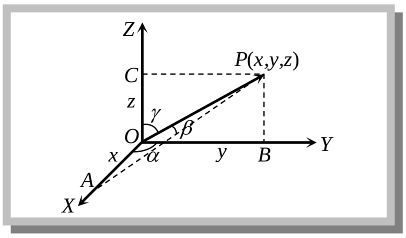
Thus, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

By definition, it follows that the direction cosine of the axis of *x* are respectively $\cos 0^o$, $\cos 90^o$, $\cos 90^o$ *i.e.* (1, 0, 0). Similarly direction cosines of the axes of *y* and *z* are respectively (0, 1, 0) and (0, 0, 1).

Relation between the direction cosines : Let *OP* be any line through the origin *O* which has direction cosines *l*, *m*, *n*. Let *P* = (*x*, *y*, *z*) and *OP* = *r*. Then $OP^2 = x^2 + y^2 + z^2 = r^2$ .....(i)

From *P* draw *PA*, *PB*, *PC* perpendicular on the co-ordinate axes, so that

OA = *x*, *OB* = *y*, *OC* = z. Also, $\angle POA = \alpha, \angle POB = \beta$ and $\angle POC = \gamma$.



From triangle *AOP*, $l = \cos\alpha = \frac{x}{r} \Rightarrow x = lr$

Similarly $y = mr$ and $z = nr$.

Hence from (i), $r^2(l^2 + m^2 + n^2) = x^2 + y^2 + z^2 = r^2 \Rightarrow l^2 + m^2 + n^2 = 1$

or, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, or, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Note : ❑ If *OP* = *r* and the co-ordinates of point *P* be (*x*, *y*, *z*), then d.c.'s of line *OP* are *x*/*r*, *y*/*r*, *z*/*r*.

❑ Direction cosines of $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are $\frac{a}{|\mathbf{r}|}, \frac{b}{|\mathbf{r}|}, \frac{c}{|\mathbf{r}|}$.

❑ Since $-1 \le \cos x \le 1$, $\forall x \in R$, hence values of *l*, *m*, *n* are such real numbers which are not less than – 1 and not greater than 1. Hence $\text{d.c.'s} \in [-1, 1]$.

❑ The direction cosines of a line parallel to any co-ordinate axis are equal to the direction cosines of the co-ordinate axis.

❑ The number of lines which are equally inclined to the co-ordinate axes is 4.

❑ If *l*, *m*, *n* are the d.c.'s of a line, then the maximum value of $lmn = \frac{1}{3\sqrt{3}}$.

Important Tips

☞ *The angles α, β, γ are called the direction angles of line AB.*

☞ *The d.c.'s of line BA are cos (π – α), cos (π – β) and cos (π – γ) i.e., –cosα, –cosβ, –cosγ.*

☞ *Angles α, β, γ are not coplanar.*

☞ *α + β + γ is not equal to 360° as these angles do not lie in same plane.*

☞ *If P(x, y,z) be a point in space such that* $\mathbf{r} = \overrightarrow{OP}$ *has d.c.'s l, m, n then* $x = l|\mathbf{r}|$, $y = m|\mathbf{r}|$, $z = n|\mathbf{r}|$.

☞ *Projection of a vector* **r** *on the co-ordinate axes are* $l|\mathbf{r}|$, $m|\mathbf{r}|$, $n|\mathbf{r}|$.

☞ $\mathbf{r} = |\mathbf{r}|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$ *and* $\hat{\mathbf{r}} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$

(2) **Direction ratio**

(i) Three numbers which are proportional to the direction cosines of a line are called the direction ratio of that line. If *a*, *b*, *c* are three numbers proportional to direction cosines *l*, *m*, *n* of a line, then *a*, *b*, *c* are called its direction ratios. They are also called direction numbers or direction components.

Hence by definition, we have $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (say) $\Rightarrow l = ak, m = bk, n = ck$

$\Rightarrow l^2 + m^2 + n^2 = (a^2 + b^2 + c^2) = k^2 \Rightarrow k = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$

$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}$, $m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$, $n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$

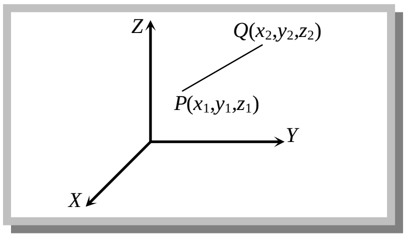
where the sign should be taken all positive or all negative.

Note : ❑ Direction ratios are not uniques, whereas d.c.'s are unique. *i.e.,* $a^2 + b^2 + c^2 \ne 1$

(ii) Let $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a vector. Then its d.r.'s are *a*, *b*, *c*

If a vector **r** has d.r.'s *a, b, c* then $\mathbf{r}=\frac{|\mathbf{r}|}{\sqrt{a^2+b^2+c^2}}(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$

(iii) **D.c.'s and d.r.'s of a line joining two points :** The direction ratios of line *PQ* joining $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are $x_2-x_1=a$, $y_2-y_1=b$ and $z_2-z_1=c$ (say).



Then direction cosines are,

$$l=\frac{(x_2-x_1)}{\sqrt{\Sigma(x_2-x_1)^2}},m=\frac{(y_2-y_1)}{\sqrt{\Sigma(x_2-x_1)^2}},n=\frac{(z_2-z_1)}{\sqrt{\Sigma(x_2-x_1)^2}}$$

i.e., $l=\frac{x_2-x_1}{PQ},m=\frac{y_2-y_1}{PQ},n=\frac{z_2-z_1}{PQ}$.

Example: 7 A line makes the same angle θ with each of the *x* and *z*-axis. If the angle β, which it makes with *y*-axis, is such that $\sin^2\beta=3\sin^2\theta$, then $\cos^2\theta$ equals **[AIEEE 2004]**

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{1}{5}$ (d) $\frac{2}{3}$

Solution: (b) We know that, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. Since line makes angle θ with *x* and *z*-axis and angle β with *y*-axis.

$\Rightarrow \cos^2\theta+\cos^2\beta+\cos^2\theta=1 \Rightarrow -(2\cos^2\theta-1)=\cos^2\beta$ ……(i)

Given that $\sin^2\beta=3\sin^2\theta$ ……(ii)

From (i) and (ii), $1=3\sin^2\theta-2\cos^2\theta+1 \Rightarrow 0=3(1-\cos^2\theta)-2\cos^2\theta \Rightarrow 5\cos^2\theta=3 \Rightarrow \cos^2\theta=3/5$

Example: 8 Direction cosines of the line that makes equal angles with the three axes in a space are **[Kurukshetra CEE 1995]**

(a) $\pm\frac{1}{3},\pm\frac{1}{3},\pm\frac{1}{3}$ (b) $\pm\frac{6}{7},\pm\frac{2}{3},\pm\frac{3}{7}$ (c) $\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}$ (d) $\pm\sqrt{\frac{1}{7}},\pm\sqrt{\frac{3}{14}},\pm\sqrt{\frac{1}{14}}$

Solution: (c) $\because l^2+m^2+n^2=1 \Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

Now, $\alpha=\beta=\gamma$

$\Rightarrow 3\cos^2\alpha=1 \Rightarrow \cos\alpha=\pm1/\sqrt{3}$ *i.e.,* $l=m=n=\pm1/\sqrt{3}$.

Hence required d.c.'s are $\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}$.

Example: 9 A line which makes angle 60° with *y*-axis and *z*-axis, then the angle which it makes with *x*-axis is **[Rajasthan PET 2002; DCE 1996]**

(a) 45° (b) 60° (c) 75° (d) 30°

Solution: (a) Given that $\beta=\gamma=60°$ *i.e.* $m=\cos\beta=\cos 60°=1/2$, $n=\cos\gamma=\cos 60°=1/2$

$\because l^2+m^2+n^2=1 \Rightarrow l^2=1-\frac{1}{4}-\frac{1}{4}=\frac{1}{2} \Rightarrow l=\frac{1}{\sqrt{2}} \Rightarrow \cos\alpha=\frac{1}{\sqrt{2}} \Rightarrow \alpha=45°$

Example: 10 A line passes through the points (6, –7, –1) and (2, –3, 1). The direction cosines of line, so directed that the angle made by it with the positive direction of *x*-axis is acute, are

(a) $\frac{2}{3},\frac{-2}{3},\frac{-1}{3}$ (b) $\frac{-2}{3},\frac{2}{3},\frac{1}{3}$ (c) $\frac{2}{3},\frac{-2}{3},\frac{1}{3}$ (d) $\frac{2}{3},\frac{2}{3},\frac{1}{3}$

Solution: (a) Let *l, m, n* be the d.c.'s of a given line.

Then, as it makes an acute angle with *x*-axis, therefore *l*>0.

Direction ratios = 4, –4, –2 or 2, –2, –1 and Direction cosines $=\frac{2}{3},\frac{-2}{3},\frac{-1}{3}$.

Example: 11 If the direction cosines of a line are $\left(\frac{1}{c},\frac{1}{c},\frac{1}{c}\right)$, then **[DCE 2000; Pb. CET 1996, 98]**

(a) $c > 0$ (b) $c = \pm\sqrt{3}$ (c) $0 < c < 1$ (d) $c > 2$

Solution: (b) We know that $l^2 + m^2 + n^2 = 1 \Rightarrow \frac{1}{c^2} + \frac{1}{c^2} + \frac{1}{c^2} = 1 \Rightarrow \frac{3}{c^2} = 1 \Rightarrow c = \pm\sqrt{3}$.

Example: 12 If **r** is a vector of magnitude 21 and has d.r.'s 2, –3, 6. Then **r** is equal to

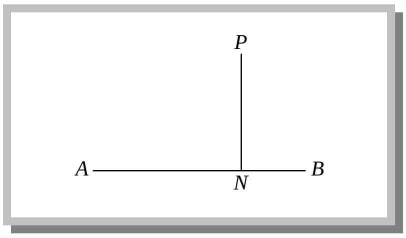
(a) $6\mathbf{i} - 9\mathbf{j} + 18\mathbf{k}$ (b) $6\mathbf{i} + 9\mathbf{j} + 18\mathbf{k}$ (c) $6\mathbf{i} - 9\mathbf{j} - 18\mathbf{k}$ (d) $6\mathbf{i} + 9\mathbf{j} - 18\mathbf{k}$

Solution: (a) D.r.'s of **r** are 2, –3, 6. Therefore, its d.c.'s are $l = \frac{2}{7}, m = \frac{-3}{7}, n = \frac{6}{7}$

$$\therefore\ \mathbf{r} = |\mathbf{r}|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k}) = 21\left[\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right] = 6\mathbf{i} - 9\mathbf{j} + 18\mathbf{k}.$$

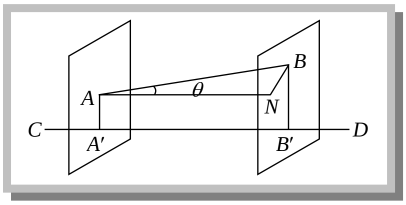
7.7 Projection

(1) **Projection of a point on a line :** The projection of a point P on a line AB is the foot N of the perpendicular PN from P on the line AB.



N is also the same point where the line AB meets the plane through P and perpendicular to AB.

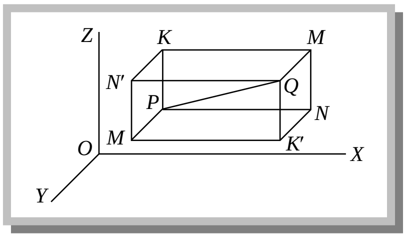
(2) **Projection of a segment of a line on another line and its length :** The projection of the segment AB of a given line on another line CD is the segment $A'B'$ of CD where A' and B' are the projections of the points A and B on the line CD.

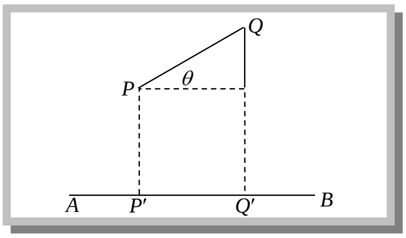


The length of the projection $A'\,B'$.

$$A'B' = AN = AB\cos\theta$$

(3) **Projection of a line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m and n :** Let PQ be a line segment where $P \equiv (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ and AB be a given line with d.c.'s as l, m, n. If the line segment PQ makes angle θ with the line AB, then





Projection of PQ is $P'Q' = PQ\cos\theta = (x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma$

$$= (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$$

Important Tips

- *For x-axis, $l = 1$, $m = 0$, $n = 0$.*
  Hence, projection of PQ on x-axis = $x_2 - x_1$, Projection of PQ on y-axis = $y_2 - y_1$ and Projection of PQ on z-axis = $z_2 - z_1$
- *If P is a point (x_1, y_1, z_1), then projection of OP on a line whose direction cosines are l, m, n, is $l_1x_1 + m_1y_1 + n_1z_1$, where O is the origin.*
- *If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c.'s of two concurrent lines, then the d.c.'s of the lines bisecting the angles between them are proportional to $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$.*

Example: 13 If A, B, C, D are the points (3, 4, 5) (4, 6, 3), (–1, 2, 4) and (1, 0, 5), then the projection of CD on AB is

[Orissa JEE 2002; Rajasthan PET 2002]

(a) $\frac{3}{4}$ (b) $\frac{-4}{3}$ (c) $\frac{3}{5}$ (d) None of these

Solution: (b) Let l, m, n be the direction cosines of AB

Then $l = \frac{4-3}{\sqrt{(4-3)^2+(6-4)^2+(3-5)^2}} = \frac{1}{3}$, $m = \frac{6-4}{3} = \frac{2}{3}$. Similarly $n = \frac{-2}{3}$

$\therefore$ The projection of CD on $AB = \left[1-(-1)\left(\frac{1}{3}\right)\right] + [0-2]\left(\frac{2}{3}\right) + [5-4]\left(-\frac{2}{3}\right) = \frac{2}{3} - \frac{4}{3} + \left(-\frac{2}{3}\right) = -\frac{4}{3}$

Example: 14 The projection of a line on co-ordinate axes are 2, 3, 6. Then the length of the line is **[Orissa JEE 2002]**

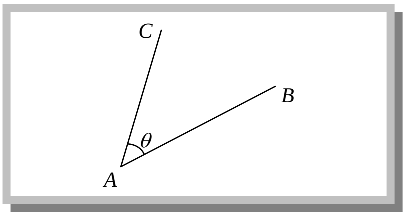
(a) 7 (b) 5 (c) 1 (d) 11

Solution: (b) Let AB be the line and its direction cosines be $\cos\alpha$, $\cos\beta$, $\cos\gamma$. Then the projection of line AB on the co-ordinate axes are $AB\cos\alpha$, $AB\cos\beta$, $AB\cos\gamma$. $\therefore$ $AB\cos\alpha = 2$, $AB\cos\beta = 3$, $AB\cos\gamma = 6$

$\Rightarrow AB^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2^2+3^2+6^2 = 49 \Rightarrow AB^2(1) = 49 \Rightarrow AB = 7$

7.8 Angle between Two lines

(1) **Cartesian form :** Let θ be the angle between two straight lines AB and AC whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, is given by $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$.



If direction ratios of two lines a_1, b_1, c_1 and a_2, b_2, c_2 are given, then angle between two lines is given by $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,.\sqrt{a_2^2+b_2^2+c_2^2}}$.

Particular results : We have, $\sin^2\theta = 1-\cos^2\theta = (l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2) - (l_1l_2+m_1m_2+n_1n_2)^2$

$$= (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2$$

$\Rightarrow \sin\theta = \pm\sqrt{\Sigma(l_1m_2 - l_2m_1)^2}$, which is known as Lagrange's identity.

The value of $\sin\theta$ can easily be obtained by the following form. $\sin\theta = \sqrt{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2 + \begin{vmatrix} m_1 & n_1 \\ n_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2}$

When d.r.'s of the lines are given if a_1, b_1, c_1 and a_2, b_2, c_2 are d.r.'s of given two lines, then angle θ between them is given by $\sin\theta = \frac{\sqrt{\Sigma(a_1b_2 - a_2b_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

Condition of perpendicularity : If the given lines are perpendicular, then $\theta = 90°$ *i.e.* $\cos\theta = 0$

$\Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Condition of parallelism : If the given lines are parallel, then $\theta = 0^o$ *i.e.* $\sin\theta = 0$

$\Rightarrow (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 = 0$, which is true, only when

$l_1m_2 - l_2m_1 = 0$, $m_1n_2 - m_2n_1 = 0$ and $n_1l_2 - n_2l_1 = 0$

$\Rightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

Similarly, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Note : ❑ The angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

❑ The angle between a diagonal of a cube and the diagonal of a faces of the cube is $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$.

❑ If a straight line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

❑ If the edges of a rectangular parallelopiped be a, b, c, then the angles between the two diagonals are $\cos^{-1}\left[\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right]$

(2) **Vector form :** Let the vector equations of two lines be $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \lambda \mathbf{b}_2$

As the lines are parallel to the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ respectively, therefore angle between the lines is same as the angle between the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. Thus if θ is the angle between the given lines, then $\cos\theta = \frac{\mathbf{b}_1.\mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$.

Note : ❑ If the lines are perpendicular, then $\mathbf{b}_1.\mathbf{b}_2 = 0$.

❑ If the lines are parallel, then $\mathbf{b}_1$ and $\mathbf{b}_2$ are parallel, therefore $\mathbf{b}_1 = \lambda \mathbf{b}_2$ for some scalar λ.

Example: 15 If d.c.'s of two lines are proportional to (2, 3, –6) and (3, –4,5), then the acute angle between them is **[MP PET 2003]**

(a) $\cos^{-1}\left(\frac{49}{36}\right)$ (b) $\cos^{-1}\left(\frac{18\sqrt{2}}{35}\right)$ (c) 90° (d) $\cos^{-1}\left(\frac{18}{35}\right)$

Solution: (b) D.c.'s of two lines are proportional to (2, 3, –6) and (3, – 4, 5)

i.e. d.r.'s are (2, 3, –6) and (3, –4, 5)

$$\therefore \cos\theta = \frac{2(3)+3(-4)+(-6)5}{\sqrt{2^2+3^2+(-6)^2}\sqrt{3^2+(-4)^2+5^2}} = \frac{6-12-30}{\sqrt{49}.\sqrt{50}} = \frac{-36}{7.5\sqrt{2}} \Rightarrow \cos\theta = \frac{-18\sqrt{2}}{35}$$

Taking acute angle, $\theta = \cos^{-1}\left(\frac{18\sqrt{2}}{35}\right)$

Example: 16 If the direction ratio of two lines are given by $3lm - 4ln + mn = 0$ and $l + 2m + 3n = 0$, then the angle between the lines is **[EAMCET 2003]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Solution: (a) We have, $l + 2m + 3n = 0$ ……(i)

$3lm - 4ln + mn = 0$ ……(ii)

From equation (i), $l = -(2m + 3n)$

Putting the value of l in equation (ii)

$\Rightarrow 3(-2m-3n)m + mn - 4(-2m-3n)n = 0 \Rightarrow -6m^2 - 9mn + mn + 8mn + 12n^2 = 0 \Rightarrow 6m^2 - 12n^2 = 0$

$\Rightarrow m^2 - 2n^2 = 0 \Rightarrow m + \sqrt{2}n = 0$ or $m - \sqrt{2}n = 0$

$l + 2m + 3n = 0$ ……(i) $0.l + m + \sqrt{2}n = 0$ ……(iii) $0.l + m - \sqrt{2}n = 0$ ……(iv)

From equation (i) and equation (iii), $\frac{l}{2\sqrt{2}-3} = \frac{m}{-\sqrt{2}} = \frac{n}{1}$

From equation (i) and equation (iv), $\frac{l}{-2\sqrt{2}-3} = \frac{m}{\sqrt{2}} = \frac{n}{1}$

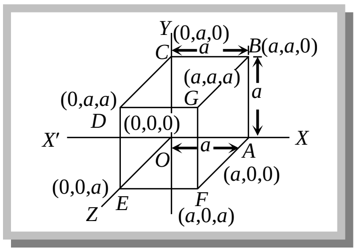
Thus, the direction ratios of two lines are $2\sqrt{2}-3, -\sqrt{2}, 1$ and $-2\sqrt{2}-3, \sqrt{2}, 1$

$(l_1, m_1, n_1) = (2\sqrt{2}-3, -\sqrt{2}, 1)$, $(l_2, m_2, n_2) = (-2\sqrt{2}-3, \sqrt{2}, 1)$, $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ .Hence, the angle between them $\pi/2$.

Example: 17 If a line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta$ is

(a) $\frac{4}{3}$ (b) 1 (c) $\frac{8}{3}$ (d) $\frac{7}{3}$

Solution: (c) Let side of the cube = a



Then OG, BE and AD, CF will be four diagonals.

d.r.'s of $OG = a, a, a = 1, 1, 1$

d.r.'s of $BE = -a, -a, a = 1, 1, -1$

d.r.'s of $AD = -a, a, a = -1, 1, 1$

d.r.'s of $CF = a, -a, a = 1, -1, 1$

Let d.r.'s of line be l, m, n. Therefore angle between line and diagonal

$$\cos\alpha = \frac{l+m+n}{\sqrt{3}}, \cos\beta = \frac{l+m-n}{\sqrt{3}}, \cos\gamma = \frac{-l+m+n}{\sqrt{3}}, \cos\delta = \frac{l-m+n}{\sqrt{3}}$$

$$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}[(l+m+n)^2 + (l+m-n)^2 + (-l+m+n)^2 + (l-m+n)^2] = \frac{4}{3}$$

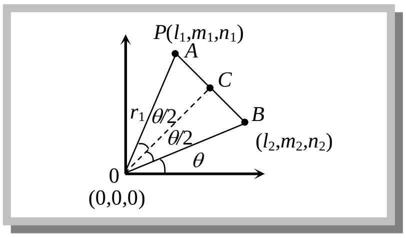
$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = \frac{8}{3}$$

Example: 18 If l_1, m_1, n_1 and l_2, m_2, n_2 are d.c.'s of two lines inclined to each other at an angle θ, then the d.c.'s of the internal bisectors of angle between these lines are

(a) $\frac{l_1+l_2}{2\sin\theta/2}, \frac{m_1+m_2}{2\sin\theta/2}, \frac{n_1+n_2}{2\sin\theta/2}$ (b) $\frac{l_1+l_2}{2\cos\theta/2}, \frac{m_1+m_2}{2\cos\theta/2}, \frac{n_1+n_2}{2\cos\theta/2}$

(c) $\frac{l_1-l_2}{2\sin\theta/2}, \frac{m_1-m_2}{2\sin\theta/2}, \frac{n_1-n_2}{2\sin\theta/2}$ (d) $\frac{l_1-l_2}{2\cos\theta/2}, \frac{m_1-m_2}{2\cos\theta/2}, \frac{n_1-n_2}{2\cos\theta/2}$

Solution: (b) Let OA and OB be two lines.



D.c.'s of OA is (l_1, m_1, n_1) and OB is (l_2, m_2, n_2).

Let $OA = OB = 1$.

Then the co-ordinates of A and B are (l_1, m_1, n_1) and (l_2, m_2, n_2).

Let OC be the bisector of $\angle AOB$.

Then C is the mid-point of AB and so its co-ordinates are $\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right)$.

$\therefore$ D.r.'s of line OC are $\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right)$

We have, $OC = \sqrt{\left(\frac{l_1+l_2}{2}\right)^2 + \left(\frac{m_1+m_2}{2}\right)^2 + \left(\frac{n_1+n_2}{2}\right)^2} = \frac{1}{2}\sqrt{l_1^2+m_1^2+n_1^2+l_2^2+m_2^2+n_2^2+2(l_1l_2+m_1m_2+n_1n_2)}$

$$= \frac{1}{2}\sqrt{1+1+2\cos\theta} = \frac{1}{2}\sqrt{2(2.\cos^2\theta/2)} = \cos\theta/2\,.$$

D.r.'s of line OC are $\frac{l_1+l_2}{2\cos\theta/2}, \frac{m_1+m_2}{2\cos\theta/2}, \frac{n_1+n_2}{2\cos\theta/2}$.

Example: 19 The angle between the lines $\mathbf{r} = (4\mathbf{i}-\mathbf{j}) + s(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$ and $\mathbf{r} = (\mathbf{i}-\mathbf{j}+2\mathbf{k}) + t(\mathbf{i}-3\mathbf{j}+2\mathbf{k})$ is **[Tamilnadu (Engg.) 2002]**

(a) $\frac{3\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{6}$

Solution: (b) We have, $\mathbf{r} = (4\mathbf{i}-\mathbf{j}) + s(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$ and $\mathbf{r} = (\mathbf{i}-\mathbf{j}+2\mathbf{k}) + t(\mathbf{i}-3\mathbf{j}+2\mathbf{k})$

We know that, $\cos\theta = \frac{\mathbf{b}_1.\mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$, $\cos\theta = \frac{(2\mathbf{i}+\mathbf{j}-3\mathbf{k}).(\mathbf{i}-3\mathbf{j}+2\mathbf{k})}{\sqrt{4+1+9}\sqrt{1+9+4}} = \frac{2-3-6}{\sqrt{14}.\sqrt{14}} = \frac{-7}{14}$, $\cos\theta = -\frac{1}{2}$

Hence, acute angle $\theta = \cos^{-1}\left(\frac{1}{2}\right)$ *i.e.* $\theta = \frac{\pi}{3}$

The Straight Line

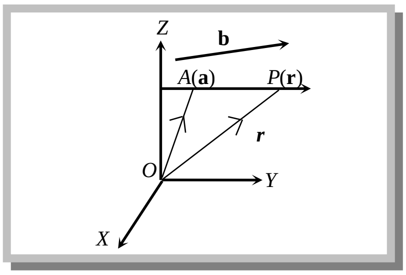
7.9 Straight line in Space

Every equation of the first degree represents a plane. Two equations of the first degree are satisfied by the co-ordinates of every point on the line of intersection of the planes represented by them. Therefore, the two equations together represent that line. Therefore $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ together represent a straight line.

(1) Equation of a line passing through a given point

(i) **Cartesian form or symmetrical form :** Cartesian equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios *a*, *b*, *c* is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

(ii) **Vector form :** Vector equation of a straight line passing through a fixed point with position vector **a** and parallel to a given vector **b** is $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$.



Important Tips

- *The parametric equations of the line* $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ *are* $x = x_1 + a\lambda, y = y_1 + b\lambda, z = z_1 + c\lambda$, *where* λ *is the parameter.*
- *The co-ordinates of any point on the line* $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ *are* $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, *where* $\lambda \in R$.
- *Since the direction cosines of a line are also direction ratios, therefore equation of a line passing through* (x_1, y_1, z_1) *and having direction cosines l, m, n is* $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$.
- *Since x, y and z-axes pass through the origin and have direction cosines 1, 0, 0; 0, 1, 0 and 0, 0, 1 respectively. Therefore, the equations are x-axis :* $\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0}$ *or y = 0 and z = 0.*

  y-axis : $\frac{x - 0}{0} = \frac{y - 0}{1} = \frac{z - 0}{0}$ *or x = 0 and z = 0; z-axis :* $\frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1}$ *or x = 0 and y = 0.*
- *In the symmetrical form of equation of a line, the coefficients of x, y, z are unity.*

7.10 Equation of Line passing through Two given points

(i) **Cartesian form :** If $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ be two given points, the equations to the line *AB* are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

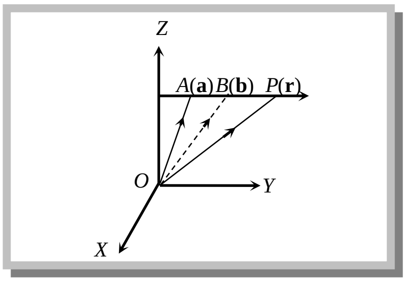
The co-ordinates of a variable point on *AB* can be expressed in terms of a parameter λ in the form

$$x = \frac{\lambda x_2 + x_1}{\lambda + 1}, y = \frac{\lambda y_2 + y_1}{\lambda + 1}, z = \frac{\lambda z_2 + z_1}{\lambda + 1}$$

λ being any real number different from –1. In fact, (*x*, *y*, *z*) are the co-ordinates of the point which divides the join of *A* and *B* in the ratio $\lambda : 1$.

(ii) **Vector form :** The vector equation of a line passing through two points with position vectors **a** and **b** is

$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$



7.11 Changing Unsymmetrical form to Symmetrical form

The unsymmetrical form of a line $ax + by + cz + d = 0, a'x + b'y + c'z + d' = 0$

Can be changed to symmetrical form as follows : $\dfrac{x - \dfrac{bd' - b'd}{ab' - a'b}}{bc' - b'c} = \dfrac{y - \dfrac{da' - d'a}{ab' - a'b}}{ca' - c'a} = \dfrac{z}{ab' - a'b}$

Example: 20 The equation to the straight line passing through the points (4, –5, –2) and (–1, 5, 3) is **[MP PET 2003]**

(a) $\frac{x-4}{1} = \frac{y+5}{-2} = \frac{z+2}{-1}$ (b) $\frac{x+1}{1} = \frac{y-5}{2} = \frac{z-3}{-1}$ (c) $\frac{x}{-1} = \frac{y}{5} = \frac{z}{3}$ (d) $\frac{x}{4} = \frac{y}{-5} = \frac{z}{-2}$

Solution: (a) We know that equation of a straight line is of the form $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

D.r.'s of the line $= (-1-4, 5+5, 3+2)$ *i.e.*, $(-5, 10, 5)$ or $(-1, 2, 1)$.

Hence the equation is $\frac{x-4}{-1} = \frac{y+5}{2} = \frac{z+2}{1}$ *i.e.*, $\frac{x-4}{1} = \frac{y+5}{-2} = \frac{z+2}{-1}$

Example: 21 The d.c.'s of the line $6x - 2 = 3y + 1 = 2z - 2$ are

(a) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ (c) 1, 2, 3 (d) None of these

Solution: (b) We have $6x - 2 = 3y + 1 = 2z - 2 \Rightarrow \frac{6x - (2/6)}{1} = \frac{3y + (1/3)}{1} = \frac{2(z-1)}{1}$

$\Rightarrow \frac{x - (1/3)}{1/6} = \frac{y + (1/3)}{1/3} = \frac{z-1}{1/2} \Rightarrow \frac{x - (1/3)}{1} = \frac{y + (1/3)}{2} = \frac{z-1}{3}$

d.r.'s of line are (1, 2, 3). Hence d.c.'s of line are $(1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$

Example: 22 The vector equation of line through the point *A*(3, 4, –7) and *B*(1, –1, 6) is **[Pb. CET 1999]**

(a) $\mathbf{r} = (3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + 6\mathbf{k})$ (b) $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 6\mathbf{k}) + \lambda(3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k})$

(c) $\mathbf{r} = (3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) + \lambda(-2\mathbf{i} - 5\mathbf{j} + 13\mathbf{k})$ (d) $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 6\mathbf{k}) + \lambda(4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

Solution: (c) Position vector of *A* is $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ and that of *B* is $\mathbf{b} = \mathbf{i} - \mathbf{j} + 6\mathbf{k}$

We know that equation of line in vector form, $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, $\mathbf{r} = (3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) + \lambda(-2\mathbf{i} - 5\mathbf{j} + 13\mathbf{k})$.

7.12 Angle between Two lines

Let the cartesian equations of the two lines be

$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ .....(i) and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ .....(ii)

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition of perpendicularity : If the lines are perpendicular, then $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

Condition of parallelism : If the lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Example: 23 If the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$ are at right angles, then $k=$ **[MP PET 1997, 2001; DCE 1997, 99]**

(a) –10 (b) 10/7 (c) –10/7 (d) –7/10

Solution: (a) We have $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$

Since lines are $\perp$ to each other. So, $a_1a_2+b_1b_2+c_1c_2=0$

$(-3)(3k)+(2k)(1)+(2)(-5)=0 \Rightarrow -9k+2k-10=0 \Rightarrow -7k=10 \Rightarrow k=-10/7$.

Example: 24 The lines $x=ay+b$, $z=cy+d$ and $x=a'y+b'$, $z=c'y+d'$ are perpendicular to each other, if **[IIT 1984; AIEEE 2003]**

(a) $aa'+cc'=1$ (b) $aa'+cc'=-1$ (c) $ac+a'c'=1$ (d) $ac+a'c'=-1$

Solution: (b) We have, $x=ay+b$, $z=cy+d$

$\frac{x-b}{a}=y$, $\frac{z-d}{c}=y \Rightarrow \frac{x-b}{a}=\frac{y-0}{1}=\frac{z-d}{c}$ ……(i)

and $x=a'y+b'$, $z=c'y+d'$

$\frac{x-b'}{a'}=y$, $\frac{z-d'}{c'}=y \Rightarrow \frac{x-b'}{a'}=\frac{y-0}{1}=\frac{z-d'}{c'}$ ……(ii)

$\because$ Given, lines (i) and (ii) are perpendicular

$\therefore$ $a(a')+1(1)+c(c')=0$, $aa'+cc'=-1$

Example: 25 The direction ratio of the line which is perpendicular to the lines $\frac{x-7}{2}=\frac{y+17}{-3}=\frac{z-6}{1}$ and $\frac{x+5}{1}=\frac{y+3}{2}=\frac{z-4}{-2}$ are **[Pb. CET 1999]**

(a) < 4, 5, 7 > (b) < 4, –5, 7 > (c) < 4, –5, –7 > (d) < –4, 5, 7 >

Solution: (a) Let d.r.'s of line be *l*, *m*, *n*.

$\because$ line is perpendicular to given line

$\therefore$ $2l-3m+n=0$ ……(i)

$l+2m-2n=0$ ……(ii)

From equation (i) and (ii)

$\frac{l}{6-2}=\frac{m}{1+4}=\frac{n}{4+3}$ or $\frac{l}{4}=\frac{m}{5}=\frac{n}{7}$. Hence, d.r.'s of line (< 4, 5, 7 >)

7.13 Reduction of Cartesian form of the Equation of a line to Vector form and Vice versa

Cartesian to vector : Let the Cartesian equation of a line be $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ ……(i)

This is the equation of a line passing through the point $A(x_1,y_1,z_1)$ and having direction ratios *a*, *b*, *c*. In vector form this means that the line passes through point having position vector $\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and is parallel to the vector $\mathbf{m}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Thus, the vector form of (*i*) is $\mathbf{r}=\mathbf{a}+\lambda\mathbf{m}$ or $\mathbf{r}=(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})+\lambda(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$, where λ is a parameter.

Vector to cartesian : Let the vector equation of a line be $\mathbf{r}=\mathbf{a}+\lambda\mathbf{m}$ ……(ii)

Where $\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}, \mathbf{m}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ and λ is a parameter.

To reduce (ii) to Cartesian form we put $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and equate the coefficients of **i**, **j** and **k** as discussed below.

Putting $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}, \mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and $\mathbf{m}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ in (ii), we obtain

$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})+\lambda(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$

Equating coefficients of **i**, **j** and **k**, we get $x=x_1+a\lambda, y=y_1+b\lambda, z=z_1+c\lambda$ or $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda$

Example: 26 The cartesian equations of a line are $6x-2=3y+1=2z-2$. The vector equation of the line is

(a) $\mathbf{r}=\left(\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\mathbf{k}\right)+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ (b) $\mathbf{r}=(3\mathbf{i}-3\mathbf{j}+\mathbf{k})+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

(c) $\mathbf{r}=(\mathbf{i}+\mathbf{j}+\mathbf{k})+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ (d) None of these

Solution: (a) The given line is $6x-2=3y+1=2z-2 \Rightarrow \frac{x-1/3}{1}=\frac{y+1/3}{2}=\frac{z-1}{3}$

This show that the given line passes through (1/3, –1/3) and has direction ratio 1, 2, 3.

Position vector $\mathbf{a}=\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\mathbf{k}$ and is parallel to vector $\mathbf{b}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$. Hence, $\mathbf{r}=\left(\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\mathbf{k}\right)+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$.

7.14 Intersection of Two lines

Determine whether two lines intersect or not. In case they intersect, the following algorithm is used to find their point of intersection.

Algorithm for cartesian form : Let the two lines be $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ ……(i)

And $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ ……(ii)

Step I : Write the co-ordinates of general points on (i) and (ii). The co-ordinates of general points on (i) and (ii) are given by $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}=\lambda$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}=\mu$ respectively.

i.e., $(a_1\lambda+x_1, b_1\lambda+y_1+c_1\lambda+z_1)$ and $(a_2\mu+x_2, b_2\mu+y_2, c_2\mu+z_2)$

Step II : If the lines (i) and (ii) intersect, then they have a common point.

$a_1\lambda+x_1=a_2\mu+x_2, b_1\lambda+y_1=b_2\mu+y_2$ and $c_1\lambda+z_1=c_2\mu+z_2$.

Step III : Solve any two of the equations in λ and μ obtained in step II. If the values of λ and μ satisfy the third equation, then the lines (i) and (ii) intersect, otherwise they do not intersect.

Step IV : To obtain the co-ordinates of the point of intersection, substitute the value of λ (or μ) in the co-ordinates of general point (s) obtained in step I.

Example: 27 If the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then $k=$ **[IIT Screening 2004]**

(a) 2/9 (b) 9/2 (c) 0 (d) –1

Solution: (b) We have, $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=r_1$ (Let)

$x=2r_1+1, y=3r_1-1, z=4r_1+1$ *i.e.* point is $(2r_1+1, 3r_1-1, 4r_1+1)$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}=r_2$ (Let)

i.e. point is $(r_2+3, 2r_2+k, r_2)$.

If the lines are intersecting, then they have a common point.

$\Rightarrow 2r_1+1=r_2+3, 3r_1-1=2r_2+k, 4r_1+1=r_2$

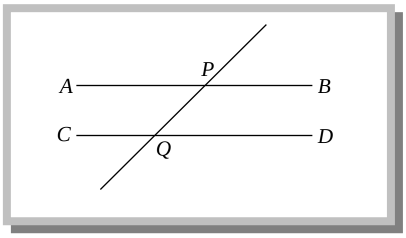
On solving, $r_1=-3/2, r_2=-5$

Hence, $k=9/2$.

Example: 28 A line with direction cosines proportional to 2, 1, 2 meets each of the lines $x=y+a=z$ and $x+a=2y=2z$. The co-ordinates of each of the points of intersection are given by **[AIEEE 2004]**

(a) (2*a*, 3*a*, 3*a*) (2*a*, *a*, *a*) (b) (3*a*, 2*a*, 3*a*) (*a*, *a*, *a*) (c) (3*a*, 2*a*, 3*a*) (*a*, *a*, 2*a*) (d) (3*a*, 3*a*, 3*a*) (*a*, *a*, *a*)

Solution: (b) Given lines are $\frac{x}{1}=\frac{y+a}{1}=\frac{z}{1}=\lambda$ (say) $\therefore$ Point is $P(\lambda, \lambda-a, \lambda)$



and $\frac{x+a}{1}=\frac{y}{1/2}=\frac{z}{1/2}$ *i.e.* $\frac{x+a}{2}=\frac{y}{1}=\frac{z}{1}=\mu$ (say)

$\therefore$ Point $Q(2\mu-a, \mu, \mu)$

Since d.r.'s of given lines are 2, 1, 2 and d.r.'s of $PQ=(2\mu-a-\lambda, \mu-\lambda+a, \mu-\lambda)$

According to question, $\frac{2\mu-a-\lambda}{2}=\frac{\mu-\lambda+a}{1}=\frac{\mu-\lambda}{2}$

Then $\lambda=3a$, $\mu=a$. Therefore, points of intersection are $P(3a, 2a, 3a)$ and $Q(a, a, a)$.

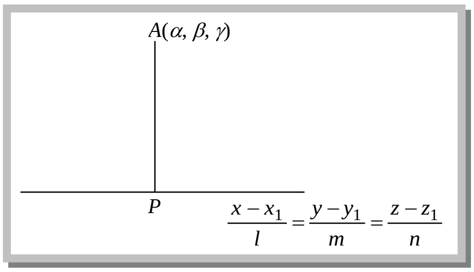
Alternative method : Check by option $x=y+a=z$ *i.e.* $3a=2a+a=3a$

$\Rightarrow a=a=a$ and $x+a=2y=2z$ *i.e.* $a+a=2a=2a \Rightarrow a=a=a$. Hence (b) is correct.

7.15 Foot of perpendicular from a point $A(\alpha, \beta, \gamma)$ to the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

(1) Cartesian form

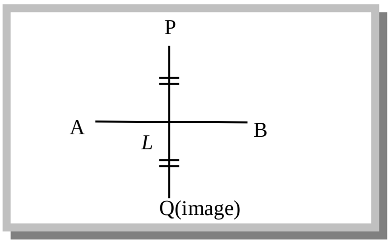
Foot of perpendicular from a point $A(\alpha, \beta, \gamma)$ to the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$: If P be the foot of perpendicular, then P is $(lr+x_1, mr+y_1, nr+z_1)$. Find the direction ratios of AP and apply the condition of perpendicularity of AP and the given line. This will give the value of r and hence the point P which is foot of perpendicular.



Length and equation of perpendicular : The length of the perpendicular is the distance AP and its equation is the line joining two known points A and P.

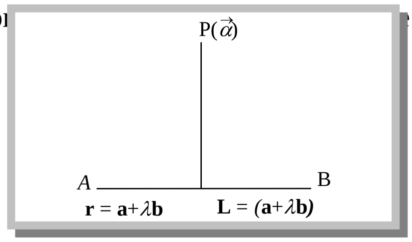
Note : ❑ The length of the perpendicular is the perpendicular distance of given point from that line.

Reflection or image of a point in a straight line : If the perpendicular PL from point P on the given line be produced to Q such that $PL=QL$, then Q is known as the image or reflection of P in the given line. Also, L is the foot of the perpendicular or the projection of P on the line.



(2) Vector form

Perpendicular distance of a point from a line : Let L is the foot of perpendicular drawn from $P(\vec{\alpha})$ on the line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. Since $\mathbf{r}$ denotes the positio
$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. So, let the position vector of L be $\mathbf{a}+\lambda\mathbf{b}$.

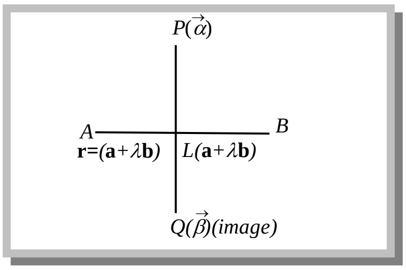


Then $\overrightarrow{PL}=\mathbf{a}-\vec{\alpha}+\lambda\mathbf{b}=(\mathbf{a}-\vec{\alpha})-\left(\frac{(\mathbf{a}-\vec{\alpha})\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$

The length PL, is the magnitude of $\overrightarrow{PL}$, and required length of perpendicular.

Image of a point in a straight line : Let $Q(\vec{\beta})$ is the image of P in $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$

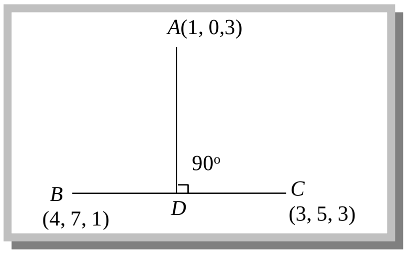
Then, $\vec{\beta} = 2\mathbf{a} - \left(\frac{2(\mathbf{a} - \vec{\alpha}).\mathbf{b}}{|\mathbf{b}|^2} \right) \mathbf{b}.\alpha$



Example: 29 The co-ordinates of the foot of the perpendicular drawn from the point $A(1, 0, 3)$ to the join of the points $B(4, 7, 1)$ and $C(3, 5, 3)$ are **[Rajasthan PET 2001]**

(a) (5/3, 7/3, 17/3) (b) (5, 7, 17) (c) (5/3, –7/3, 17/3) (d) (–5/3, 7/3, –17/3)

Solution: (a) Equation of BC, $\frac{x-4}{-1} = \frac{y-7}{-2} = \frac{z-1}{2}$



i.e. $\frac{x-4}{1} = \frac{y-7}{2} = \frac{z-1}{-2} = r$ (say)

Any point on the given line is $D(r+4, 2r+7, -2r+1)$

Then, d.r.'s of AD = $(r+4-1, 2r+7-0, -2r+1-3)$

i.e. d.r.'s of $AD = (r+3, 2r+7, -2r-2)$ and d.r.'s of BC = (–1, –2, 2)

Since AD is $\perp$ to given line,

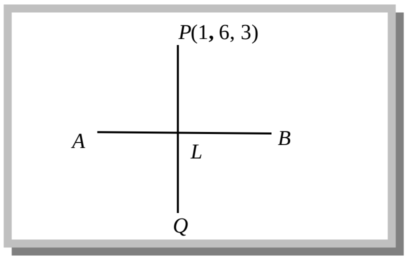
$\therefore\ (-1)(r+3)+(2r+7)(-2)+(2)(-2r-2)=0 \Rightarrow -r-3-4r-14-4r-4=0 \Rightarrow -9r-21=0 \Rightarrow r=-7/3$

$\therefore$ D is {4 – (7/3), 7– (14/3), (14/3)+1} *i.e.* D is (5/3, 7/3, 17/3).

Example: 30 The image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ is

(a) (1, 0, 7) (b) (–1, 0, 7) (c) (1, 0, –7) (d) None of these

Solution: (a) Let $P(1, 6, 3)$ be the given point, and let L be the foot of the perpendicular from P to the given line. The co-ordinates of a general point on the given line are given by $\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda$



i.e. $x = \lambda, y = 2\lambda+1, z = 3\lambda+2$.

Let the co-ordinates of L be $(\lambda, 2\lambda + 1, 3\lambda + 2)$ ……(i)

So, direction ratios of PL are $\lambda-1, 2\lambda+1-6, 3\lambda+2-3$ *i.e.* $\lambda-1, 2\lambda-5, 3\lambda-1$.

Direction ratios of the given line are 1, 2, 3 which is perpendicular to PL.

$\therefore\ (\lambda-1).1+(2\lambda-5)2+(3\lambda-1)3=0 \Rightarrow 14\lambda-14=0 \Rightarrow \lambda=1$

So, co-ordinates of L are (1, 3, 5). Let $Q(x_1, y_1, z_1)$ be the image of $P(1, 6, 3)$ in the given line.

Then L is the mid-point of PQ.

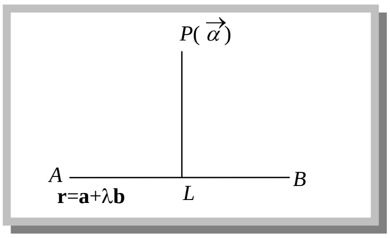
$\therefore\ \frac{x_1+1}{2} = 1, \frac{y_1+6}{2} = 3$ and $\frac{z_1+3}{2} = 5 \Rightarrow x_1 = 1, y_1 = 0$ and $z_1 = 7$.

Hence the image of $P(1, 6, 3)$ in the given line is (1, 0, 7).

Example: 31 The length of the perpendicular from the origin to line $\mathbf{r} = (4\mathbf{i}+2\mathbf{j}+4\mathbf{k}) + \lambda(3\mathbf{i}+4\mathbf{j}-5\mathbf{k})$ is **[AMU 1992]**

(a) $2\sqrt{5}$ (b) 2 (c) $5\sqrt{2}$ (d) 6

Solution: (d) $\vec{\alpha} = 0.\mathbf{i} + 0.\mathbf{j} + 0.\mathbf{k}$



$$\overrightarrow{PL} = (\mathbf{a} - \vec{\alpha}) - \left(\frac{(\mathbf{a} - \vec{\alpha}).\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$$

$$\overrightarrow{PL} = (4\mathbf{i}+2\mathbf{j}+4\mathbf{k}) - \left[\frac{(4\mathbf{i}+2\mathbf{j}+4\mathbf{k}).(3\mathbf{i}+4\mathbf{j}-5\mathbf{k})}{9+16+25}\right](3\mathbf{i}+4\mathbf{j}-5\mathbf{k}) = 4\mathbf{i}+2\mathbf{j}+4\mathbf{k} - \left(\frac{12+8-20}{50}\right).(3\mathbf{i}+4\mathbf{j}-5\mathbf{k})$$

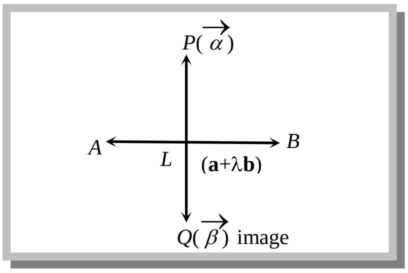
$$\overrightarrow{PL} = 4\mathbf{i}+2\mathbf{j}+4\mathbf{k}$$

The length of PL is magnitude of $\overrightarrow{PL}$ *i.e.*, Length of perpendicular $= |\overrightarrow{PL}| = \sqrt{16+4+16} = 6$.

Example: 32 The image of point (1, 2, 3) in the line $\mathbf{r} = (6\mathbf{i}+7\mathbf{j}+7\mathbf{k}) + \lambda(3\mathbf{i}+2\mathbf{j}-2\mathbf{k})$ is

(a) (5, –8, 15) (b) (5, 8, –15) (c) (–5, –8, –15)

Solution: (d) Given that, $\mathbf{a} = 6\mathbf{i}+7\mathbf{j}+7\mathbf{k}$, $\mathbf{b} = 3\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $\vec{\alpha} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$



Then, $\vec{\beta} = 2\mathbf{a} - \left(\frac{2(\mathbf{a} - \vec{\alpha}).\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} - \vec{\alpha}$

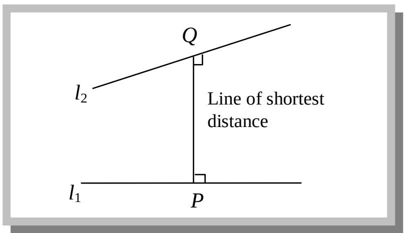
$$= 2(6\mathbf{i}+7\mathbf{j}+7\mathbf{k}) - \left(\frac{2(5\mathbf{i}+5\mathbf{j}+4\mathbf{k}).(3\mathbf{i}+2\mathbf{j}-2\mathbf{k})}{9+4+4}\right)(3\mathbf{i}+2\mathbf{j}-2\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+3\mathbf{k})$$

On solving, $\vec{\beta} = 5\mathbf{i}+8\mathbf{j}+15\mathbf{k}$. Thus $\vec{\beta}$ is the position vector of Q, which is the image of P in given line.

Hence image of point (1, 2, 3) in the given line is (5, 8, 15).

7.16 Shortest distance between two straight lines

(1) **Skew lines :** Two straight lines in space which are neither parallel nor intersecting are called skew lines. Thus, the skew lines are those lines which do not lie in the same plane.



(2) **Line of shortest distance :** If l_1 and l_2 are two skew lines, then the straight line which is perpendicular to each of these two non-intersecting lines is called the "line of shortest distance."

Note : ❑ There is one and only one line perpendicular to each of lines l_1 and l_2.

(3) **Shortest distance between two skew lines**

(i) **Cartesian form :** Let two skew lines be $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$

Therefore, the shortest distance between the lines is given by

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - l_1n_2)^2 + (l_1m_2 - l_2m_1)^2}}$$

(ii) **Vector form :** Let l_1 and l_2 be two lines whose equations are $l_1 : \mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $l_2 : \mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2$ respectively. Then, Shortest distance $PQ = \left| \frac{(\mathbf{b}_1 \times \mathbf{b}_2).(\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|} \right| = \frac{|[\mathbf{b}_1 \, \mathbf{b}_2 \, (\mathbf{a}_2 - \mathbf{a}_1)]}{|\mathbf{b}_1 \times \mathbf{b}_2|}$

(4) **Shortest distance between two parallel lines :** The shortest distance between the parallel lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}$ is given by $d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{b}|}$.

(5) **Condition for two lines to be intersecting *i.e.* coplanar**

(i) **Cartesian form :** If the lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ intersect, then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 .$$

(ii) **Vector form :** If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \lambda \mathbf{b}_2$ intersect, then the shortest distance between them is zero. Therefore, $[\mathbf{b}_1 \mathbf{b}_2 (\mathbf{a}_2 - \mathbf{a}_1)] = 0 \Rightarrow [(\mathbf{a}_2 - \mathbf{a}_1) \; \mathbf{b}_1 \mathbf{b}_2] = 0 \Rightarrow (\mathbf{a}_2 - \mathbf{a}_1).(\mathbf{b}_1 \times \mathbf{b}_2) = 0$

Important Tips

- *Skew lines are non-coplanar lines.*
- *Parallel lines are not skew lines.*
- *If two lines intersect, the shortest distance (SD) between them is zero.*
- *Length of shortest distance between two lines is always taken to be positive.*
- *Shortest distance between two skew lines is perpendicular to both the lines.*

(6) **To determine the equation of line of shortest distance :** To find the equation of line of shortest distance, we use the following procedure :

(i) From the given equations of the straight lines,

i.e. $\frac{x - a_1}{l_1} = \frac{y - b_1}{m_1} = \frac{z - c_1}{n_1} = \lambda$ (say) ……(i)

and $\frac{x - a_2}{l_2} = \frac{y - b_2}{m_2} = \frac{z - c_2}{n_2} = \mu$ (say) ……(ii)

Find the co-ordinates of general points on straight lines (i) and (ii) as

$(a_1 + \lambda l_1, b_1 + \lambda m_1, c_1 + \lambda n_1)$ and $(a_2 + \mu l_2, b_2 + \mu m_2, c_2 + \mu n_2)$.

(ii) Let these be the co-ordinates of P and Q, the two extremities of the length of shortest distance. Hence, find the direction ratios of PQ as $(a_2 + l_2\mu) - (a_1 + l_1\lambda), (b_2 + m_2\mu) - (b_1 + m_1\lambda), (c_2 + m_2\mu) - (c_1 + n_1\lambda)$.

(iii) Apply the condition of PQ being perpendicular to straight lines (i) and (ii) in succession and get two equations connecting λ and μ. Solve these equations to get the values of λ and μ.

(iv) Put these values of λ and μ in the co-ordinates of P and Q to determine points P and Q.

(v) Find out the equation of the line passing through P and Q, which will be the line of shortest distance.

Note : ❑ The same algorithm may be observed to find out the position vector of P and Q, the two extremities of the shortest distance, in case of vector equations of straight lines. Hence, the line of shortest distance, which passes through P and Q, can be obtained.

Example: 33 The shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$ is **[Kerala (Engg.)2001; DCE 1993]**

(a) $\frac{1}{6}$ (b) $\frac{1}{\sqrt{6}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{3}$

Solution: (b) S.D. $=\dfrac{\begin{vmatrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}}{\sqrt{(15-16)^2+(12-10)^2+(8-9)^2}}=\dfrac{\begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}}{\sqrt{1+1+4}}=\frac{1}{\sqrt{6}}$.

Example: 34 The shortest distance between the lines $\mathbf{r}=(\mathbf{i}+\mathbf{j}-\mathbf{k})+\lambda(3\mathbf{i}-\mathbf{j})$ and $\mathbf{r}=(4\mathbf{i}-\mathbf{k})+\mu(2\mathbf{i}+3\mathbf{k})$ is **[Pb. CET 1995]**

(a) 6 (b) 0 (c) 2 (d) 4

Solution: (b) S.D. $=\left|\frac{(\mathbf{b}_1\times\mathbf{b}_2).(\mathbf{a}_2-\mathbf{a}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|=\left|\frac{[(3\mathbf{i}-\mathbf{j})\times(2\mathbf{i}+3\mathbf{k})].(3\mathbf{i}-\mathbf{j})}{|(3\mathbf{i}-\mathbf{j})\times(2\mathbf{i}+3\mathbf{k})|}\right|=\left|\frac{(-3\mathbf{i}-9\mathbf{j}+2\mathbf{k}).(3\mathbf{i}-\mathbf{j})}{\sqrt{9+81+4}}\right|=\frac{-9+9+0}{\sqrt{94}}$.

Hence, S.D. = 0

Example: 35 The line $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar, if **[AIEEE 2003]**

(a) $k = 0$ or –1 (b) $k = 0$ or 1 (c) $k = 0$ or –3 (d) $k = 3$ or –3

Solution: (c) Lines are coplanar, if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} 1-2 & 4-3 & 5-4 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix}=0 \Rightarrow k^2+3k=0 \Rightarrow k(k+3)=0 \Rightarrow k=0,\ k=-3$$

Example: 36 The lines $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}\times\mathbf{c})$ and $\mathbf{r}=\mathbf{b}+\mu(\mathbf{c}\times\mathbf{a})$ will intersect if

(a) $\mathbf{a}\times\mathbf{c}=\mathbf{b}\times\mathbf{c}$ (b) $\mathbf{a}.\mathbf{c}=\mathbf{b}.\mathbf{c}$ (c) $\mathbf{b}\times\mathbf{a}=\mathbf{c}\times\mathbf{a}$ (d) None of these

Solution: (b) If lines are intersecting, then

$(\mathbf{a}_2-\mathbf{a}_1).(\mathbf{b}_1\times\mathbf{b}_2)=0 \Rightarrow \mathbf{b}\,(\mathbf{a}-\mathbf{b}).[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})]=0$

$\Rightarrow (\mathbf{a}-\mathbf{b}).[(\mathbf{b}\times\mathbf{c}.\mathbf{a})\mathbf{c}-(\mathbf{b}\times\mathbf{c}.\mathbf{c})\mathbf{a}]=0 \Rightarrow (\mathbf{a}-\mathbf{b})[(\mathbf{b}\times\mathbf{c}.\mathbf{a})\mathbf{c}]=0$

$\Rightarrow [(\mathbf{a}-\mathbf{b}).\mathbf{c}]\mathbf{abc}=0 \Rightarrow (\mathbf{a}.\mathbf{c}-\mathbf{b}.\mathbf{c})(\mathbf{abc})=0 \Rightarrow \mathbf{a}.\mathbf{c}-\mathbf{bc}=0 \Rightarrow \mathbf{a}.\mathbf{c}=\mathbf{b}.\mathbf{c}$

Example: 37 If the straight lines $x=1+s, y=3-\lambda s, z=1+\lambda s$ and $x=\frac{t}{2}, y=1+t, z=2-t$, with parameters s and t respectively, are co-planar, then λ equals **[AIEEE 2004]**

(a) 0 (b) –1 (c) $-\frac{1}{2}$ (d) –2

Solution: (d) We have $\frac{x-1}{1}=\frac{y+3}{-\lambda}=\frac{z-1}{\lambda}=s$ and $\frac{2x}{1}=\frac{y-1}{1}=\frac{z-2}{-1}=t$

i.e. $\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-2}{-2}=\frac{t}{2}$

Since, lines are co-planar,

Then, $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} -1 & 4 & 1 \\ 1 & -\lambda & \lambda \\ 1 & 2 & -2 \end{vmatrix}=0$

On solving, $\lambda=-2$.

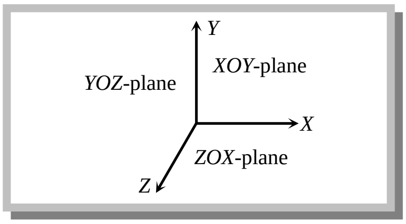
The Plane

7.17 Definition of plane and its equations

If point $P(x, y, z)$ moves according to certain rule, then it may lie in a 3-D region on a surface or on a line or it may simply be a point. Whatever we get, as the region of P after applying the rule, is called locus of P. Let us discuss about the plane or curved surface. If Q be any other point on it's locus and all points of the straight line PQ lie on it, it is a plane. In other words if the straight line PQ, however small and in whatever direction it may be, lies completely on the locus, it is a plane, otherwise any curved surface.

(1) **General equation of plane :** Every equation of first degree of the form $Ax + By + Cz + D = 0$ represents the equation of a plane. The coefficients of x, y and z *i.e.* A, B, C are the direction ratios of the normal to the plane.

(2) **Equation of co-ordinate planes**



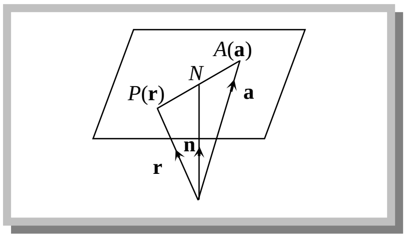
XOY-plane : $z = 0$

YOZ -plane : $x = 0$

ZOX-plane : $y = 0$

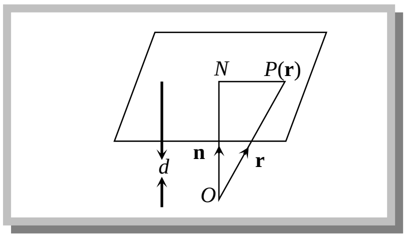
(3) **Vector equation of plane**

(i) Vector equation of a plane through the point $A(\mathbf{a})$ and perpendicular to the vector $\mathbf{n}$ is $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$ or $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$



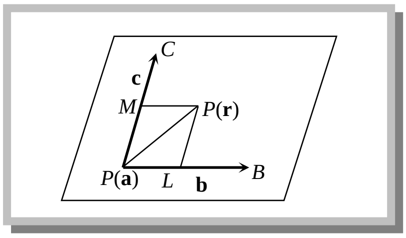
Note : ❑ The above equation can also be written as $\mathbf{r}.\mathbf{n} = d$, where $d = \mathbf{a}.\mathbf{n}$. This is known as the scalar product form of a plane.

(4) **Normal form :** Vector equation of a plane normal to unit vector $\hat{\mathbf{n}}$ and at a distance d from the origin is $\mathbf{r}.\hat{\mathbf{n}} = d$.



Note : ❑ If $\mathbf{n}$ is not a unit vector, then to reduce the equation $\mathbf{r}.\mathbf{n} = d$ to normal form we divide both sides by $|\mathbf{n}|$ to obtain $\mathbf{r} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{d}{|\mathbf{n}|}$ or $\mathbf{r}.\hat{\mathbf{n}} = \frac{d}{|\mathbf{n}|}$.

(5) **Equation of a plane passing through a given point and parallel to two given vectors :** The equation of the plane passing through a point having position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ and $\mathbf{c}$ is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where λ and μ are scalars.



(6) **Equation of plane in various forms**

(i) **Intercept form** : If the plane cuts the intercepts of length a, b, c on co-ordinate axes, then its equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(ii) **Normal form :** Normal form of the equation of plane is $lx + my + nz = p$,

where l, m, n are the d.c.'s of the normal to the plane and p is the length of perpendicular from the origin.

(7) **Equation of plane in particular cases**

(i) Equation of plane through the origin is given by $Ax + By + Cz = 0$.

i.e. if $D = 0$, then the plane passes through the origin.

(8) **Equation of plane parallel to co-ordinate planes or perpendicular to co-ordinate axes**

(i) Equation of plane parallel to *YOZ*-plane (or perpendicular to *x*-axis) and at a distance '*a*' from it is $x = a$.

(ii) Equation of plane parallel to *ZOX*-plane (or perpendicular to *y*-axis) and at a distance '*b*' from it is $y = b$.

(iii) Equation of plane parallel to *XOY*-plane (or perpendicular to *z*-axis) and at a distance '*c*' from it is $z = c$.

Important Tips

☞ *Any plane perpendicular to co-ordinate axis is evidently parallel to co-ordinate plane and vice versa.*

☞ *A unit vector perpendicular to the plane containing three points A, B, C is* $\dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$.

(9) **Equation of plane perpendicular to co-ordinate planes or parallel to co-ordinate axes**

(i) Equation of plane perpendicular to *YOZ*-plane or parallel to *x*-axis is $By + Cz + D = 0$.

(ii) Equation of plane perpendicular to *ZOX*-plane or parallel to *y* axis is $Ax + Cz + D = 0$.

(iii) Equation of plane perpendicular to *XOY*-plane or parallel to *z*-axis is $Ax + By + D = 0$.

(10) **Equation of plane passing through the intersection of two planes**

(i) **Cartesian form :** Equation of plane through the intersection of two planes $P = a_1x + b_1y + c_1z + d_1 = 0$ and $Q = a_2x + b_2y + c_2z + d_2 = 0$ is $P + \lambda Q = 0$, where λ is the parameter.

(ii) **Vector form :** The equation of any plane through the intersection of planes $\mathbf{r}.\mathbf{n}_1 = d_1$ and $\mathbf{r}.\mathbf{n}_2 = d_2$ is $\mathbf{r}.(\mathbf{n}_1 + \lambda\mathbf{n}_2) = d_1 + \lambda d_2$, where λ is an arbitrary constant.

(11) **Equation of plane parallel to a given plane**

(i) **Cartesian form :** Plane parallel to a given plane $ax + by + cz + d = 0$ is $ax + by + cz + d' = 0$, *i.e.* only constant term is changed.

(ii) **Vector form :** Since parallel planes have the common normal, therefore equation of plane parallel to plane $\mathbf{r}.\mathbf{n} = d_1$ is $\mathbf{r}.\mathbf{n} = d_2$, where d_2 is a constant determined by the given condition.

7.18 Equation of plane passing through the given point

(1) **Equation of plane passing through a given point :** Equation of plane passing through the point (x_1, y_1, z_1) is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, where *A*, *B* and *C* are d.r.'s of normal to the plane.

(2) **Equation of plane through three points :** The equation of plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$ or $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$.

7.19 Foot of perpendicular from a point $A(\alpha, \beta, \gamma)$ to a given plane $ax + by + cz + d = 0$

If *AP* be the perpendicular from *A* to the given plane, then it is parallel to the normal, so that its equation is

$$\frac{x - \alpha}{a} = \frac{y - \beta}{b} = \frac{z - \gamma}{c} = r \qquad \text{(say)}$$

Any point *P* on it is $(ar + \alpha, br + \beta, cr + \gamma)$. It lies on the given plane and we find the value of *r* and hence the point *P*.

(1) **Perpendicular distance**

(i) **Cartesian form :** The length of the perpendicular from the point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is $\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$.

Note : ❑ The distance between two parallel planes is the algebraic difference of perpendicular distances on the planes from origin.

❑ Distance between two parallel planes $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$ is $\frac{D_2 \sim D_1}{\sqrt{A^2 + B^2 + C^2}}$.

(ii) **Vector form :** The perpendicular distance of a point having position vector **a** from the plane $\mathbf{r}.\mathbf{n} = d$ is given by $p = \frac{|\mathbf{a}.\mathbf{n} - d|}{|\mathbf{n}|}$

(2) **Position of two points *w.r.t.* a plane :** Two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs. The plane divides the line joining the points P and Q externally or internally according to P and Q are lying on same or opposite sides of the plane.

7.20 Angle between two planes

(1) **Cartesian form :** Angle between the planes is defined as angle between normals to the planes drawn from any point. Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}\right)$$

Note : ❑ If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then the planes are perpendicular to each other.

❑ If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the planes are parallel to each other.

(2) **Vector form :** An angle θ between the planes $\mathbf{r}_1.\mathbf{n}_1 = d_1$ and $\mathbf{r}_2.\mathbf{n}_2 = d_2$ is given by $\cos\theta = \pm\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$.

7.21 Equation of planes bisecting angle between two given planes

(1) **Cartesian form :** Equations of planes bisecting angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d = 0$ are $\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$.

Note : ❑ If angle between bisector plane and one of the plane is less than 45°, then it is acute angle bisector, otherwise it is obtuse angle bisector.

- If $a_1a_2 + b_1b_2 + c_1c_2$ is negative, then origin lies in the acute angle between the given planes provided d_1 and d_2 are of same sign and if $a_1a_2 + b_1b_2 + c_1c_2$ is positive, then origin lies in the obtuse angle between the given planes.

(2) **Vector form :** The equation of the planes bisecting the angles between the planes $\mathbf{r}_1.\mathbf{n}_1 = d_1$ and $\mathbf{r}_2.\mathbf{n}_2 = d_2$ are $\frac{|\mathbf{r}.\mathbf{n}_1 - d_1|}{|\mathbf{n}_1|} = \frac{|\mathbf{r}.\mathbf{n}_2 - d_2|}{|\mathbf{n}_2|}$ or $\frac{\mathbf{r}.\mathbf{n}_1 - d_1}{|\mathbf{n}_1|} = \pm\frac{\mathbf{r}.\mathbf{n}_2 - d_2}{|\mathbf{n}_2|}$ or $\mathbf{r}.(\hat{\mathbf{n}}_1 \pm \hat{\mathbf{n}}_2) = \frac{d_1}{|\mathbf{n}_1|} \pm \frac{d_2}{|\mathbf{n}_2|}$.

7.22 Image of a point in a plane

Let P and Q be two points and let π be a plane such that

(i) Line PQ is perpendicular to the plane π, and

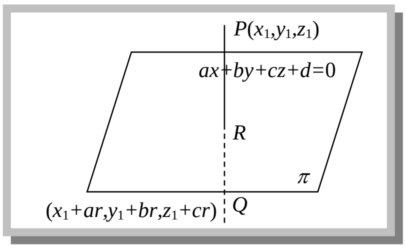
(ii) Mid-point of PQ lies on the plane π.

Then either of the point is the image of the other in the plane π.

To find the image of a point in a given plane, we proceed as follows

(i) Write the equations of the line passing through P and normal to the given plane as

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$



(ii) Write the co-ordinates of image Q as $(x_1 + ar, y_1, + br, z_1 + cr)$.

(iii) Find the co-ordinates of the mid-point R of PQ.

(iv) Obtain the value of r by putting the co-ordinates of R in the equation of the plane.

(v) Put the value of r in the co-ordinates of Q.

7.23 Coplanar lines

Lines are said to be coplanar if they lie in the same plane or a plane can be made to pass through them.

(1) **Condition for the lines to be coplanar**

(i) **Cartesian form :** If the lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are coplanar

Then $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

The equation of the plane containing them is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$ or $\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

(ii) **Vector form :** If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \lambda\mathbf{b}_2$ are coplanar, then $[\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2] = [\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2]$ and the equation of the plane containing them is $[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2] = [\mathbf{a}_1\,\mathbf{b}_1\,\mathbf{b}_2]$ or $[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2] = [\mathbf{a}_2\,\mathbf{b}_1\,\mathbf{b}_2]$.

Note :
- Every pair of parallel lines is coplanar.

❑ Two coplanar lines are either parallel or intersecting.

❑ The three sides of a triangle are coplanar.

Important Tips

☞ ***Division by plane :*** *The ratio in which the line segment PQ, joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2), is divided by plane* $ax + by + cz + d = 0$ *is* $= -\left(\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right)$.

☞ ***Division by co-ordinate planes :*** *The ratio in which the line segment PQ, joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) is divided by co-ordinate planes are as follows* :

(i) By yz-plane : $-x_1/x_2$ *(ii) By zx-plane* : $-y_1/y_2$ *(ii) By xy-plane* : $-z_1/z_2$

Example: 38 The *xy*-plane divides the line joining the points (–1, 3, 4) and (2, –5, 6) **[Rajasthan PET 2000]**

(a) Internally in the ratio 2 : 3 (b) Internally in the ratio 3 : 2

(c) Externally in the ratio 2 : 3 (d) Externally in the ratio 3 : 2

Solution: (c) Required ratio $= -\dfrac{z_1}{z_2} = -\left(\dfrac{4}{6}\right) = -\dfrac{2}{3}$

$\therefore$ *xy*-plane divide externally in the ratio 2 : 3.

Example: 39 The ratio in which the plane $x - 2y + 3z = 17$ divides the line joining the point (–2, 4, 7) and (3, –5, 8) is **[AISSE 1988]**

(a) 10 : 3 (b) 3 : 1 (c) 3 : 10 (d) 10 : 1

Solution: (c) Required ratio $= -\left(\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right) = -\left(\dfrac{-2 - 8 + 21 - 17}{3 + 10 + 24 - 17}\right) = \dfrac{6}{20} = \dfrac{3}{10}$.

Example: 40 The equation of the plane, which makes with co-ordinate axes a triangle with its centroid (α, β, γ), is **[MP PET 2004]**

(a) $\alpha x + \beta y + \gamma z = 3$ (b) $\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 1$ (c) $\alpha x + \beta y + \gamma z = 1$ (d) $\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 3$

Solution: (d) We know that $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ ……(i)

Centroid $\left(\dfrac{a}{3}, \dfrac{b}{3}, \dfrac{c}{3}\right)$ *i.e.* $\alpha = a/3, \beta = b/3, \gamma = c/3 \Rightarrow a = 3\alpha, b = 3\beta, c = 3\gamma$

From equation (i), $\dfrac{x}{3\alpha} + \dfrac{y}{3\beta} + \dfrac{z}{3\gamma} = 1$

$\therefore \dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 3$.

Example: 41 The equation of plane passing through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x + 6y + 6z = 1$ is **[AISSE 1984; Tamilnadu (Engg.) 2002]**

(a) $3x + 4y + 5z = 9$ (b) $3x + 4y + 5z = 0$ (c) $3x + 4y - 5z = 9$ (d) None of these

Solution: (c) We know that, equation of plane is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

It passes through (2, 2, 1)

$\therefore a(x - 2) + b(y - 2) + c(z - 1) = 0$ ……(i)

Plane (i) also passes through (9, 3, 6) and is perpendicular to the plane $2x + 6y + 6z = 1$

$\therefore 7a + b + 5c = 0$ ……(ii)

and $2a + 6b + 6c = 0$ ……(iii)

$\dfrac{a}{6 - 30} = \dfrac{b}{10 - 42} = \dfrac{c}{42 - 2}$ or $\dfrac{a}{-24} = \dfrac{b}{-32} = \dfrac{c}{40}$

or $\frac{a}{3} = \frac{b}{4} = \frac{c}{-5} = k$ (say)

From equation (i), $3k(x-2)+4k(y-2)+(-5)k(z-1)=0$

Hence, $3x+4y-5z=9$.

Example: 42 The equation of the plane containing the line $\mathbf{r}=\mathbf{a}+k\mathbf{b}$ and perpendicular to the plane $\mathbf{r}.\mathbf{n}=q$ is

(a) $(\mathbf{r}-\mathbf{b}).(\mathbf{n}\times\mathbf{a})=0$ (b) $(\mathbf{r}-\mathbf{a}).(\mathbf{n}\times(\mathbf{a}\times\mathbf{b}))=0$ (c) $(\mathbf{r}-\mathbf{a}).(\mathbf{n}\times\mathbf{b})=0$ (d) $(\mathbf{r}-\mathbf{b}).(\mathbf{n}\times(\mathbf{a}\times\mathbf{b}))=0$

Solution: (c) Since the required plane contains the line $\mathbf{r}=\mathbf{a}+k\mathbf{b}$ and is perpendicular to the plane $\mathbf{r}.\mathbf{n}=q$.

$\therefore$ It passes through the point **a** and parallel to vectors **b** and **n**. Hence, it is perpendicular to the vector $\mathbf{N}=\mathbf{n}\times\mathbf{b}$.

$\therefore$ Equation of the required plane is $(\mathbf{r}-\mathbf{a}).\mathbf{N}=0 \Rightarrow (\mathbf{r}-\mathbf{a}).(\mathbf{n}\times\mathbf{b})=0$.

Example: 43 The equation of the plane through the intersection of the planes $x+2y+3z-4=0$, $4x+3y+2z+1=0$ and passing through the origin will be **[MP PET 1997; Kerala (Engg.) 2001; AISSE 1983]**

(a) $x+y+z=0$ (b) $17x+14y+11z=0$ (c) $7x+4y+z=0$ (d) $17x+14y+z=0$

Solution: (b) Any plane through the given planes is $(x+2y+3z-4)+k(4x+3y+2z+1)=0$

It passes through (0, 0, 0)

$\therefore\ -4+k=0 = k=4$

$\therefore$ Required plane is $(x+2y+3z-4)+4(4x+3y+2z+1)=0 \Rightarrow 17x+14y+11z=0$.

Example: 44 The vector equation of the plane passing through the origin and the line of intersection of plane $\mathbf{r}.\mathbf{a}=\lambda$ and $\mathbf{r}.\mathbf{b}=\mu$ is

(a) $\mathbf{r}.(\lambda\mathbf{a}-\mu\mathbf{b})=0$ (b) $\mathbf{r}.(\lambda\mathbf{b}-\mu\mathbf{a})=0$ (c) $\mathbf{r}.(\lambda\mathbf{a}+\mu\mathbf{b})=0$ (d) $\mathbf{r}.(\lambda\mathbf{b}+\mu\mathbf{a})=0$

Solution: (b) The equation of a plane through the line of intersection of plane $\mathbf{r}.\mathbf{a}=\lambda$ and $\mathbf{r}.\mathbf{b}=\mu$ can be written as $\mathbf{r}.(\mathbf{a}+k\mathbf{b})=\lambda+k\mu$ ……(i)

This passes through the origin, therefore putting the value of k in (i),

$\mathbf{r}(\mu\mathbf{a}-\lambda\mathbf{b})=0 \Rightarrow \mathbf{r}.(\lambda\mathbf{b}-\mu\mathbf{a})=0$.

Example: 45 Angle between two planes $x+2y+2z=3$ and $-5x+3y+4z=9$ is **[IIT Screening 2004]**

(a) $\cos^{-1}\frac{3\sqrt{2}}{10}$ (b) $\cos^{-1}\frac{19\sqrt{2}}{30}$ (c) $\cos^{-1}\frac{9\sqrt{2}}{20}$ (d) $\cos^{-1}\frac{3\sqrt{2}}{5}$

Solution: (a) We know that, $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{1(-5)+2(3)+2(4)}{\sqrt{1+4+4}\sqrt{25+9+16}} = \frac{9}{3.5\sqrt{2}} = \frac{3\sqrt{2}}{10}$

i.e. $\theta=\cos^{-1}\left(\frac{3\sqrt{2}}{10}\right)$.

Example: 46 Distance between two parallel planes $2x+y+2z=8$ and $4x+2y+4z+5=0$ is **[AIEEE 2004]**

(a) $\frac{9}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{3}{2}$

Solution: (c) We have $2x+y+2z-8=0$ ……(i)

and $4x+2y+4z+5=0$ or $2x+y+2z+5/2=0$ ……(ii)

Distance between the planes $=\frac{(5/2)+8}{\sqrt{4+1+4}} = \frac{21}{2.3} = \frac{7}{2}$.

Example: 47 A tetrahedron has vertices at $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then the angle between the faces OAB and ABC will be **[MNR 1994; UPSEAT 2000; AIEEE 2003]**

(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{17}{31}\right)$ (c) 30° (d) 90°

Solution: (a) Angle between two plane faces is equal to the angle between the normals n_1 and n_2 to the planes. $\mathbf{n_1}$, the normal to the face

OAB is given by $\overrightarrow{OA}\times\overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\mathbf{i}-\mathbf{j}-3\mathbf{k}$ ……(i)

$\mathbf{n_2}$, the normal to the face ABC, is given by $\overrightarrow{AB}\times\overrightarrow{AC}$.

$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \mathbf{i}-5\mathbf{j}-3\mathbf{k}$ ……(ii)

If θ be the angle between $\mathbf{n_1}$ and $\mathbf{n_2}$, Then $\cos\theta = \frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{5.1+5+9}{\sqrt{35}\sqrt{35}}$

$\cos\theta = \frac{19}{35} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{35}\right)$.

Example: 48 The distance of the point (2, 1, –1) from the plane $x - 2y + 4z = 9$ is **[Kerala (Engg.) 2001]**

(a) $\frac{\sqrt{13}}{21}$ (b) $\frac{13}{21}$ (c) $\frac{13}{\sqrt{21}}$ (d) $\sqrt{\frac{13}{21}}$

Solution: (c) Distance of the plane from (2, 1, –1) $= \left|\frac{2-2(1)+4(-1)-9}{\sqrt{1+4+16}}\right| = \frac{13}{\sqrt{21}}$.

Example: 49 A unit vector perpendicular to plane determined by the points P(1, –1, 2), Q(2, 0, –1) and R(0, 2, 1) is **[IIT 1994]**

(a) $\frac{2\mathbf{i}-\mathbf{j}+\mathbf{k}}{\sqrt{6}}$ (b) $\frac{2\mathbf{i}+\mathbf{j}+k}{\sqrt{6}}$ (c) $\frac{-2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{6}}$ (d) $\frac{2\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{6}}$

Solution: (b) We know that, $\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{|\overrightarrow{PQ}\times\overrightarrow{PR}|}$

$\overrightarrow{PQ} = \mathbf{i}+\mathbf{j}-3\mathbf{k}$, $\overrightarrow{PR} = -\mathbf{i}+3\mathbf{j}-\mathbf{k}$

$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ and $|\overrightarrow{PQ}\times\overrightarrow{PR}| = 4\sqrt{6}$

Hence, the unit vector is $\frac{4(2\mathbf{i}+\mathbf{j}+\mathbf{k})}{4\sqrt{6}}$ *i.e.* $\frac{2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{6}}$.

Example: 50 The perpendicular distance from origin to the plane through the point (2, 3, –1) and perpendicular to vector $3\mathbf{i}-4\mathbf{j}+7\mathbf{k}$ is

(a) $\frac{13}{\sqrt{74}}$ (b) $-\frac{13}{\sqrt{74}}$ (c) 13 (d) None of these

Solution: (a) We know, the equation of the plane is $(\mathbf{r}-\mathbf{a}).\mathbf{n} = 0$

or $(\mathbf{r}-(2\mathbf{i}+3\mathbf{j}-\mathbf{k})).(3\mathbf{i}-4\mathbf{j}+7\mathbf{k}) = 0 \Rightarrow (x\mathbf{i}+y\mathbf{j}+z\mathbf{k}-2\mathbf{i}-3\mathbf{j}+\mathbf{k}).(3\mathbf{i}-4\mathbf{j}+7\mathbf{k}) = 0 \Rightarrow 3x-4y+7z+13 = 0$

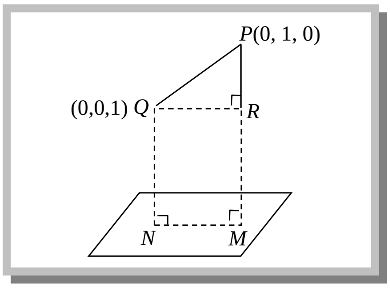
Hence, perpendicular distance of the plane from origin $= \frac{13}{\sqrt{3^2+(-4)^2+7^2}} = \frac{13}{\sqrt{74}}$.

Example: 51 If P = (0, 1, 0), Q =(0, 0, 1), then projection of PQ on the plane $x+y+z=3$ is **[EAMCET 2002]**

(a) $\sqrt{3}$ (b) 3 (c) $\sqrt{2}$ (d) 2

Solution: (c) Given plane is $x+y+z-3=0$. From point P and Q draw PM and QN perpendicular on the given plane and $QR \perp MP$.

$|MP| = \left|\frac{0+1+0-3}{\sqrt{1^2+1^2+1^2}}\right| = \frac{2}{\sqrt{3}}$



$|NQ| = \frac{2}{\sqrt{3}}$

$|PQ| = \sqrt{(0-0)^2+(0-1)^2+(1-0)^2} = \sqrt{2}$

$|RP| = |MP|-|MR| = |MP|-|NQ| = 0$ (*i.e.* R and P are the same point)

$\therefore |NM| = |QR| = \sqrt{PQ^2 - RP^2} = \sqrt{(\sqrt{2})^2 - 0} = \sqrt{2}$

Example: 52 The reflection of the point (2, –1, 3) in the plane $3x-2y-z=9$ is **[AMU 1995]**

(a) $\left(\frac{26}{7},\frac{15}{7},\frac{17}{7}\right)$ (b) $\left(\frac{26}{7},\frac{-15}{7},\frac{17}{7}\right)$ (c) $\left(\frac{15}{7},\frac{26}{7},\frac{-17}{7}\right)$ (d) $\left(\frac{26}{7},\frac{17}{7},\frac{-15}{7}\right)$

Solution: (b) Let P be the point (2, –1, 3) and Q be its reflection in the given plane.

Then, PQ is perpendicular to the given plane

Hence, d.r.'s of *PQ* are 3, –2, 1 and consequently, equations of *PQ* are $\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-3}{-1}$

Any point on this line is $(3r+2,-2r-1,-r+3)$

Let this point be *Q*. Then midpoint of *PQ* $=\left(\frac{3r+2+2}{2},\frac{-2r-1-1}{2},\frac{-r+3+3}{2}\right)=\left(\frac{3r+4}{2},-r-1,\frac{-r+6}{2}\right)$

This point lies in given plane *i.e.* $3\left(\frac{3r+4}{2}\right)-2(-r-1)-\left(\frac{-r+6}{2}\right)=9 \Rightarrow 9r+12+4r+4+r-6=9 \Rightarrow 14r=8 \Rightarrow r=\frac{4}{7}$

Hence, the required point *Q* is $\left(3\left(\frac{4}{7}\right)+2,-2\left(\frac{4}{7}\right)-1,\frac{-4}{7}+3\right)=\left(\frac{26}{7},\frac{-15}{7},\frac{17}{7}\right)$.

Example: 53 A non-zero vector **a** is parallel to the line of intersection of the plane determined by the vectors **i**, **i** + **j** and the plane determined by the vectors **i** – **j**, **i** + **k**. The angle between **a** and the vector **i** – 2**j** + 2**k** is **[IIT 1996]**

(a) $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ (b) $\frac{2\pi}{4}$ or $\frac{3\pi}{4}$ (c) $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ (d) None of these

Solution: (a) Equation of plane containing **i** and **i** + **j** is

$[\mathbf{r}-\mathbf{i},\mathbf{i},\mathbf{i}+\mathbf{j}]=0 \Rightarrow (\mathbf{r}-\mathbf{i}).[\mathbf{i}\times(\mathbf{i}+\mathbf{j})]=0 \Rightarrow [(x-1)\mathbf{i}+y\mathbf{j}+z\mathbf{k}].\mathbf{k}=0 \Rightarrow z=0$ ……(i)

Equation of plane containing **i** – **j** and **i** + **k** is

$\Rightarrow [\mathbf{r}-(\mathbf{i}-\mathbf{j})\ \ \mathbf{i}-\mathbf{j}\ \ \mathbf{i}+\mathbf{k}]=0 \Rightarrow (\mathbf{r}-\mathbf{i}+\mathbf{j})[(\mathbf{i}-\mathbf{j})\times(\mathbf{i}+\mathbf{k})]=0 \Rightarrow x+y-z=0$ …… (ii)

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Since **a** is parallel to (i) and (ii)

$a_3=0$, $a_1+a_2-a_3=0 \Rightarrow a_1=-a_2$, $a_3=0$

Thus a vector in the direction of **a** is $\mathbf{u}=\mathbf{i}-\mathbf{j}$. If θ is the angle between **a** and $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$.

Then $\cos\theta=\pm\frac{1(1)+(-1)(-2)}{\sqrt{1+1}\sqrt{1+4+4}}=\pm\frac{3}{\sqrt{2}.3} \Rightarrow \cos\theta=\pm\frac{1}{\sqrt{2}} \Rightarrow \theta=\pi/4$ or $3\pi/4$

Example: 54 The d.r.'s of normal to the plane through (1, 0, 0) and (0, 1, 0) which makes an angle $\pi/4$ with plane $x+y=3$, are **[AIEEE 2002]**

(a) $1,\sqrt{2},1$ (b) $1,1,\sqrt{2}$ (c) 1, 1, 2 (d) $\sqrt{2},1,1$

Solution: (b) Let d.r.'s of normal to plane (*a*, *b*, *c*)

$a(x-1)+b(y-0)+c(z-0)=0$ ……(i)

It is passes through (0, 1, 0). $\therefore\ a+b=0 \Rightarrow b=a$. D.r.'s of normal is (*a*, *a*, *c*) and d.r.'s of given plane is (1, 1, 0)

$\therefore\ \cos\pi/4=\frac{a+a+0}{\sqrt{a^2+a^2+c^2}\sqrt{2}} \Rightarrow 4a^2=2a^2+c^2 \Rightarrow \sqrt{2}a=c$

Then, d.r.'s of normal $(a,a,\sqrt{2}a)$ or $(1,1,\sqrt{2})$.

Line and plane

7.24 Equation of plane through a given line

(1) If equation of the line is given in symmetrical form as $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, then equation of plane is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ …… (i)

where *a*, *b*, *c* are given by $al+bm+cn=0$ ……(ii)

(2) If equation of line is given in general form as $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$, then the equation of plane passing through this line is $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$.

(3) **Equation of plane through a given line parallel to another line :** Let the d.c.'s of the other line be l_2,m_2,n_2. Then, since the plane is parallel to the given line, normal is perpendicular.

$\therefore\ al_2+bm_2+cn_2=0$ ……(iii)

Hence, the plane from (i), (ii) and (iii) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

7.25 Transformation from unsymmetric form of the equation of line to the symmetric form

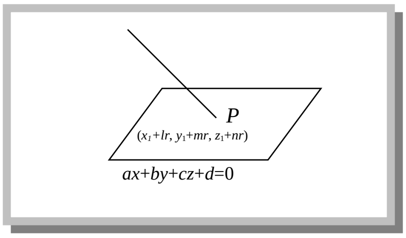
If $P \equiv a_1x + b_1y + c_1z + d_1 = 0$ and $Q \equiv a_2x + b_2y + c_2z + d_2 = 0$ are equations of two non-parallel planes, then these two equations taken together represent a line. Thus the equation of straight line can be written as $P = 0 = Q$. This form is called unsymmetrical form of a line.

To transform the equations to symmetrical form, we have to find the d.r.'s of line and co-ordinates of a point on the line.

7.26 Intersection point of a line and plane

To find the point of intersection of the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$.

The co-ordinates of any point on the line



$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ are given by

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ (say) or $(x_1 + lr, y_1 + mr, z_1 + nr)$ .....(i)

If it lies on the plane $ax + by + cz + d = 0$, then

$a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr) + d = 0 \Rightarrow (ax_1 + by_1 + cz_1 + d) + r(al + bm + cn) = 0$

$\therefore\ r = -\frac{(ax_1 + by_1 + cz_1 + d)}{al + bm + cn}$.

Substituting the value of r in (i), we obtain the co-ordinates of the required point of intersection.

Algorithm for finding the point of intersection of a line and a plane

Step I : Write the co-ordinates of any point on the line in terms of some parameters r (say).

Step II : Substitute these co-ordinates in the equation of the plane to obtain the value of r.

Step III : Put the value of r in the co-ordinates of the point in step I.

7.27 Angle between line and plane

(1) **Cartesian form :** The angle θ between the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, and the plane $ax + by + cz + d = 0$, is given by $\sin\theta = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(l^2 + m^2 + n^2)}}$.

(i) The line is perpendicular to the plane if and only if $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$.

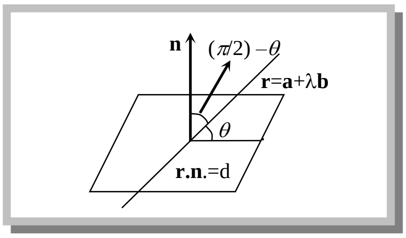
(ii) The line is parallel to the plane if and only if $al + bm + cn = 0$.

(iii) The line lies in the plane if and only if $al + bm + cn = 0$ and $a\alpha + b\beta + c\gamma + d = 0$.

(2) **Vector form :** If θ is the angle between a line $\mathbf{r} = (\mathbf{a} + \lambda\mathbf{b})$ and the plane $\mathbf{r}.\mathbf{n} = d$, then $\sin\theta = \frac{\mathbf{b}.\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$.

(i) **Condition of perpendicularity :** If the line is perpendicular to the plane, then it is parallel to the normal to the plane. Therefore **b** and **n** are parallel.

So, $\mathbf{b} \times \mathbf{n} = 0$ or $\mathbf{b} = \lambda\mathbf{n}$ for some scalar λ.



(ii) **Condition of parallelism :** If the line is parallel to the plane, then it is perpendicular to the normal to the plane. Therefore **b** and **n** are perpendicular. So, **b**.**n** = 0.

(iii) If the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ lies in the plane **r**.**n** = d, then (i) **b**.**n** = 0 and (ii) **a**.**n** = d.

7.28 Projection of a line on a plane

If P be the point of intersection of given line and plane and Q be the foot of the perpendicular from any point on the line to the plane then PQ is called the projection of given line on the given plane.

Image of line about a plane : Let line is $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$, plane is $a_2x + b_2y + c_2z + d = 0$.

Find point of intersection (say P) of line and plane. Find image (say Q) of point (x_1, y_1, z_1) about the plane. Line PQ is the reflected line.

Example: 55 The sine of angle between the straight line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and the plane $2x - 2y + z = 5$ is

[Kurukshetra CEE 1995, 2001; DCE 2000]

(a) $\frac{2\sqrt{3}}{5}$ (b) $\frac{\sqrt{2}}{10}$ (c) $\frac{4}{5\sqrt{2}}$ (d) $\frac{10}{6\sqrt{5}}$

Solution: (b) We know that $\sin\theta = \frac{al + bm + cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$

$\sin\theta = \frac{3(2)+4(-2)+5(1)}{\sqrt{9+16+25}\sqrt{4+4+1}} = \frac{3}{5\sqrt{2}.3}$

Hence, $\sin\theta = \frac{\sqrt{2}}{10}$

Example: 56 Value of k such that the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-k}{k}$ is perpendicular to normal to the plane $\mathbf{r}(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = 0$ is

[Pb. CET 2001]

(a) $-\frac{13}{4}$ (b) $-\frac{17}{4}$ (c) 4 (d) None of these

Solution: (a) We have, $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-k}{k}$

or vector form of equation of line is $\mathbf{r} = (\mathbf{i} + \mathbf{j} + k\mathbf{k}) + \lambda(2\mathbf{i} + 3\mathbf{j} + k\mathbf{k})$ *i.e.* $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + k\mathbf{k}$ and normal to the plane, $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

Given that, $\mathbf{b}.\mathbf{n} = 0$

$\Rightarrow (2\mathbf{i} + 3\mathbf{j} + k\mathbf{k}).(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = 0$

$\Rightarrow 4 + 9 + 4k = 0 \Rightarrow k = -13/4$.

Example: 57 The equation of line of intersection of the planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ can be written as **[MP PET 2004]**

(a) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{4}$ (b) $\frac{x}{2} = \frac{y}{3} = \frac{z-2}{4}$ (c) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$ (d) $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z}{4}$

Solution: (c) Let equation of line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ ……(i)

We have $4x + 4y - 5z = 12$ ……(ii) and $8x + 12y - 13z = 32$ ……(iii)

Let $z = 0$. Now putting $z = 0$ in (ii) and (iii),

we get, $4x + 4y = 12$, $8x + 12y = 32$, on solving these equations, we get $x = 1, y = 2$.

Equation of line passing through (1, 2, 0) is $\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-0}{n}$

From equation (i) and (ii),

$4l+4m-5n=0$ and $8l+12m-13n=0$

$\Rightarrow \frac{l}{8}=\frac{m}{12}=\frac{n}{16}$ *i.e.* $\frac{l}{2}=\frac{m}{3}=\frac{n}{4}$. Hence, equation of line is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}$.

Example: 58 The equation of the plane containing the two lines $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z}{3}$ and $\frac{x}{2}=\frac{y-2}{-1}=\frac{z+1}{-3}$ is **[MP PET 2000]**

(a) $8x+y-5z-7=0$ (b) $8x+y+5z-7=0$ (c) $8x-y-5z-7=0$ (d) None of these

Solution: (a) Any plane through the first line may be written as

$a(x-1)+b(y+1)+c(z)=0$ ……(i)

where, $2a-b+3c=0$ ……(ii)

It will pass through the second line, if the point (0, 2, –1) on the second line also lies on (i)

i.e. if $a(0-1)+b(2+1)+c(-1)=0$, *i.e.*, $-a+3b-c=0$ ……(iii)

Solving (ii) and (iii), we get $\frac{a}{-8}=\frac{b}{-1}=\frac{c}{5}$ *i.e.* $\frac{a}{8}=\frac{b}{1}=\frac{c}{-5}$

$\therefore$ Required plane is $8(x-1)+1(y+1)-5(z)=0 \Rightarrow 8x+y-5z-7=0$.

Example: 59 The plane which passes through the point (3, 2, 0) and the line $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$ is **[AIEEE 2002]**

(a) $x-y+z=1$ (b) $x+y+z=5$ (c) $x+2y-z=1$ (d) $2x-y+z=5$

Solution: (a) Any plane through the line $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$ is

$a(x-3)+b(y-6)+c(z-4)=0$ ……(i)

where, $a+5b+4c=0$ ……(ii)

Plane (i) passes through (3, 2, 0), if

$a(3-3)+b(2-6)+c(0-4)=0$

$-4b-4c=0$ *i.e.* $b+c=0$ ……(iii)

From equation (ii) and (iii), $a+b=0$. $\therefore a=-b=c$.

$\therefore$ Required plane is $a(x-3)-a(y-6)+a(z-4)=0$ *i.e.* $x-y+z-3+6-4=0$ *i.e.* $x-y+z=1$.

Trick : $\begin{vmatrix} x-3 & y-6 & z-4 \\ 3-3 & 2-6 & 0-4 \\ 1 & 5 & 4 \end{vmatrix}=\begin{vmatrix} x-3 & y-6 & z-4 \\ 0 & -4 & -4 \\ 1 & 5 & 4 \end{vmatrix} \Rightarrow x-y+z=1$.

Example: 60 The distance of point (–1, –5, –10) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and plane $x-y+z=5$ is**[MP PET 2**

(a) 10 (b) 8 (c) 21 (d) 13

Solution: (d) Any point on the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=r$ is $(3r+2, 4r-1, 12r+2)$

This lies on $x-y+z=5$, then $3r+2-4r+1+12r+2=5$ *i.e.* $r=0$.

$\therefore$ Point is (2, –1, 2). Its distance from (–1, –5, –10) is $\sqrt{9+16+144}=13$.

Example: 61 The value of k such that $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$ lies in the plane $2x-4y+z=7$ is **[IIT Screening 2003]**

(a) 7 (b) –7 (c) No real value (d) 4

Solution: (a) Given, point (4, 2, k) is on the line and it also passes through the plane $2x-4y+z=7 \Rightarrow 2(4)-4(2)+k=7 \Rightarrow k=7$.

Example: 62 The distance between the line $\mathbf{r}=(\mathbf{i}+\mathbf{j}+2\mathbf{k})+\lambda(2\mathbf{i}+5\mathbf{j}+3\mathbf{k})$ and the plane $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-3\mathbf{k})=5$ is **[Kurukshetra CEE 1996]**

(a) $\frac{5}{\sqrt{14}}$ (b) $\frac{6}{\sqrt{14}}$ (c) $\frac{7}{\sqrt{14}}$ (d) $\frac{8}{\sqrt{14}}$

Solution: (d) The given line is $\mathbf{r}=(\mathbf{i}+\mathbf{j}+2\mathbf{k})+\lambda(2\mathbf{i}+5\mathbf{j}+3\mathbf{k})$

$\mathbf{a}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+5\mathbf{j}+3\mathbf{k}$

Given plane, $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-3\mathbf{k})=5 \Rightarrow \mathbf{r}.\mathbf{n}=p$

Since $\mathbf{b}.\mathbf{n}=4+5-9=0$

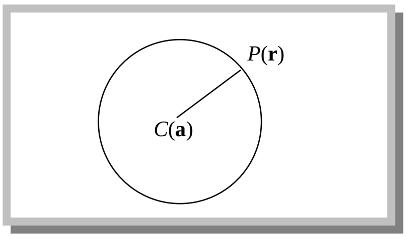
$\therefore$ The line is parallel to plane. Thus the distance between line and plane is equal to length of perpendicular from a point $\mathbf{a}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$ on line to given plane.

Hence, required distance $= \left|\frac{(\mathbf{i}+\mathbf{j}+2\mathbf{k}).(2\mathbf{i}+\mathbf{j}-3\mathbf{k})-5}{\sqrt{4+1+9}}\right| = \left|\frac{2+1-6-5}{\sqrt{14}}\right| = \frac{8}{\sqrt{14}}$.

Sphere

A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point always remains constant.

The fixed point is called the centre and the constant distance is called the radius of the sphere.



7.29 General equation of sphere

The general equation of a sphere is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ with centre (–*u*, –*v*, –*w*)

i.e. (–(1/2) coeff. of *x*, –(1/2) coeff. of *y*, –(1/2) coeff. of *z*) and, radius $=\sqrt{u^2+v^2+w^2-d}$

From the above equation, we note the following characteristics of the equation of a sphere :

(i) It is a second degree equation in *x*, *y*, *z*;

(ii) The coefficients of x^2, y^2, z^2 are all equal;

(iii) The terms containing the products *xy*, *yz* and *zx* are absent.

Note : ❑ The equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents,

(i) A real sphere, if $u^2+v^2+w^2-d>0$.

(ii) A point sphere, if $u^2+v^2+w^2-d=0$.

(iii) An imaginary sphere, if $u^2+v^2+w^2-d<0$.

Important Tips

☞ *If $u^2+v^2+w^2-d<0$, then the radius of sphere is imaginary, whereas the centre is real. Such a sphere is called "pseudo-sphere" or a "virtual sphere.*

☞ *The equation of the sphere contains four unknown constants u, v, w and d and therefore a sphere can be found to satisfy four conditions.*

7.30 Equation in sphere in various forms

(1) Equation of sphere with given centre and radius

(i) **Cartesian form :** The equation of a sphere with centre (*a*, *b*, *c*) and radius *R* is

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2 \qquad \ldots\ldots\text{(i)}$$

If the centre is at the origin, then equation (i) takes the form $x^2+y^2+z^2=R^2$,

which is known as the standard form of the equation of the sphere.

(ii) **Vector form :** The equation of sphere with centre at *C*(**c**) and radius '*a*' is $|\mathbf{r}-\mathbf{c}|=a$.

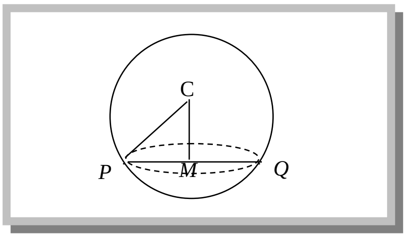
(2) Diameter form of the equation of a sphere

(i) **Cartesian form :** If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the co-ordinates of the extremities of a diameter of a sphere, then its equation is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$.

(ii) **Vector form :** If the position vectors of the extremities of a diameter of a sphere are **a** and **b**, then its equation is $(\mathbf{r}-\mathbf{a}).(\mathbf{r}-\mathbf{b})=0$ or $|\mathbf{r}|^2 - \mathbf{r}.(\mathbf{a}-\mathbf{b})+\mathbf{a}.\mathbf{b}=0$.

7.31 Section of a sphere by a plane

Consider a sphere intersected by a plane. The set of points common to both sphere and plane is called a plane section of a sphere. The plane section of a sphere is always a circle. The equations of the sphere and the plane taken together represent the plane section.



Let C be the centre of the sphere and M be the foot of the perpendicular from C on the plane. Then M is the centre of the circle and radius of the circle is given by $PM=\sqrt{CP^2-CM^2}$

The centre M of the circle is the point of intersection of the plane and line CM which passes through C and is perpendicular to the given plane.

Centre : The foot of the perpendicular from the centre of the sphere to the plane is the centre of the circle.

(radius of circle)^2 = (radius of sphere)^2 – (perpendicular from centre of spheres on the plane)^2

Great circle : The section of a sphere by a plane through the centre of the sphere is a great circle. Its centre and radius are the same as those of the given sphere.

7.32 Condition of tangency of a plane to a sphere

A plane touches a given sphere if the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere.

(1) **Cartesian form :** The plane $lx+my+nz=p$ touches the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$, if $(ul+vm+wn-p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$

(2) **Vector form :** The plane $\mathbf{r}.\mathbf{n}=d$ touches the sphere $|\mathbf{r}-\mathbf{a}|=R$ if $\frac{|\mathbf{a}.\mathbf{n}-d|}{|\mathbf{n}|}=R$.

Important Tips

☞ *Two spheres S_1 and S_2 with centres C_1 and C_2 and radii r_1 and r_2 respectively*

(i) Do not meet and lies farther apart iff $|C_1C_2|>r_1+r_2$

(ii) Touch internally iff $|C_1C_2|=|r_1-r_2|$

(iii) Touch externally iff $|C_1C_2|=r_1+r_2$

(iv) Cut in a circle iff $|r_1-r_2|<|C_1C_2|<r_1+r_2$

(v) One lies within the other if $|C_1C_2|<|r_1-r_2|$.

When two spheres touch each other the common tangent plane is $S_1-S_2=0$ and when they cut in a circle, the plane of the circle is $S_1-S_2=0$; coefficients of x^2, y^2, z^2 being unity in both the cases.

☞ *Let p be the length of perpendicular drawn from the centre of the sphere $x^2+y^2+z^2=r^2$ to the plane $Ax+By+Cz+D=0$, then*

(i) The plane cuts the sphere in a circle iff $p<r$ and in this case, the radius of circle is $\sqrt{r^2-p^2}$.

(ii) The plane touches the sphere iff $p=r$.

(iii) The plane does not meet the sphere iff $p>r$.

☞ ***Equation of concentric sphere :*** *Any sphere concentric with the sphere* $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ *is* $x^2+y^2+z^2+2ux+2vy+2wz+\lambda=0$, *where* λ *is some real which makes it a sphere.*

7.33 Intersection of straight line and a sphere

Let the equations of the sphere and the straight line be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ .....(i)

And $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ (say) .....(ii)

Any point on the line (ii) is $(\alpha+lr, \beta+mr, \gamma+nr)$.

If this point lies on the sphere (i) then we have,

$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2+2u(\alpha+lr)+2v(\beta+mr)+2w(\gamma+nr)+d=0$

or, $r^2[l^2+m^2+n^2]+2r[l(u+\alpha)+m(v+\beta)]+n(w+\gamma)]+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0$ ....(iii)

This is a quadratic equation in r and so gives two values of r and therefore the line (ii) meets the sphere (i) in two points which may be real, coincident and imaginary, according as root of (iii) are so.

Note : ❑ If l, m, n are the actual d.c.'s of the line, then $l^2+m^2+n^2=1$ and then the equation (iii) can be simplified.

7.34 Angle of intersection of two spheres

The angle of intersection of two spheres is the angle between the tangent planes to them at their point of intersection. As the radii of the spheres at this common point are normal to the tangent planes so this angle is also equal to the angle between the radii of the spheres at their point of intersection.

If the angle of intersection of two spheres is a right angle, the spheres are said to be orthogonal.

Condition for orthogonality of two spheres

Let the equation of the two spheres be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ .....(i)

and $x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$ .....(ii)

If the sphere (i) and (ii) cut orthogonally, then $2uu'+2vv'+2ww'=d+d'$, which is the required condition.

Note : ❑ If the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ cut orthogonally, then $d=a^2$.

❑ Two spheres of radii r_1 and r_2 cut orthogonally, then the radius of the common circle is $\frac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}$.

Example: 63 The centre of sphere passing through four points (0, 0, 0), (0, 2, 0), (1, 0, 0) and (0, 0, 4) is **[MP PET 2002]**

(a) $\left(\frac{1}{2},1,2\right)$ (b) $\left(-\frac{1}{2},1,2\right)$ (c) $\left(\frac{1}{2},1,-2\right)$ (d) $\left(1,\frac{1}{2},2\right)$

Solution: (a) Let the equation of sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$\because$ It passes through (0, 0, 0), $\therefore$ $d=0$

Also, It passes through (0, 2, 0) *i.e.*, $v=-1$

Also, It passes through (1, 0, 0) *i.e.*, $u = -1/2$

Also, it passes through (0, 0, 4) *i.e.*, $w = -2$

$\therefore$ Centre $(-u, -v, -w) = (1/2, 1, 1/2)$

Example: 64 The equation $|\mathbf{r}|^2 - \mathbf{r}.(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) - 10 = 0$ represents a **[DCE 1998]**

(a) Plane (b) Sphere of radius 4 (c) Sphere of radius 3 (d) None of these

Solution: (b) The given equation is $|\mathbf{r}|^2 - \mathbf{r}(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) - 10 = 0$

$\Rightarrow x^2 + y^2 + z^2 - 2x - 4y + 2z - 10 = 0$,

which is the equation of sphere, whose centre is (1, 2 –1) and radius $= \sqrt{1 + 4 + 1 + 10} = 4$.

Example: 65 The intersection of the spheres $x^2 + y^2 + z^2 + 7x - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$ is the same as the intersection of one of the sphere and the plane **[AIEEE 2004]**

(a) $2x - y - z = 1$ (b) $x - 2y - z = 1$ (c) $x - y - 2z = 1$ (d) $x - y - z = 1$

Solution: (a) We have the spheres $x^2 + y^2 + z^2 + 7x - 2y - z - 13 = 0$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z - 8 = 0$

Required plane is $S_1 - S_2 = 0$

$\therefore (7x + 3x) - (2y + 3y) - (z + 4z) - 5 = 0$

i.e. $10x - 5y + (-5z) - 5 = 0 \Rightarrow 2x - y - z = 1$.

Example: 66 The radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$ is cut by the plane $x + 2y + 2z + 7 = 0$ is **[AIEEE 2003]**

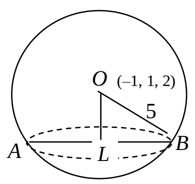
(a) 1 (b) 2 (c) 3 (d) 4

Solution: (c) For sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$, Centre O is (–1, 1, 2) and radius $= \sqrt{1 + 1 + 4 + 19} = 5$,

Now, OL = length of perpendicular from O to plane $x + 2y + 2z + 7 = 0$ is

$= \frac{-1 + 2 + 4 + 7}{\sqrt{1 + 4 + 4}} = \frac{12}{3} = 4$, *i.e.* $OL = 4$.

In ΔOLB, $LB = \sqrt{OB^2 - OL^2} = \sqrt{25 - 16} = 3$.



Example: 67 The radius of circular section of the sphere $|\mathbf{r}| = 5$ by the plane $\mathbf{r}.(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 4\sqrt{3}$ is **[DCE 1999; AMU 1991]**

(a) 2 (b) 3 (c) 4 (d) 6

Solution: (b) Radius of the sphere =5

Given plane is $x + y + z - 4\sqrt{3} = 0$

Length of the perpendicular from the centre (0, 0, 0) of the sphere to the plane $= \frac{4\sqrt{3}}{\sqrt{1 + 1 + 1}} = 4$

Hence, radius of circular section $= \sqrt{25 - 16} = 3$.

Example: 68 The shortest distance from the plane $12x + 4y + 3z = 327$ to the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z = 155$ is **[AIEEE 2003]**

(a) 26 (b) $11\frac{4}{13}$ (c) 13 (d) 39

Solution: (c) Centre of sphere is (–2, 1, 3)

Radius of sphere is $\sqrt{4 + 1 + 9 + 155} = 13$

Distance of centre from plane $= \frac{-24 + 4 + 9 - 327}{\sqrt{144 + 16 + 9}} = \frac{338}{13}$

$\therefore$ Plane cuts the sphere and hence $S.D. = \frac{338}{13} - 13 = \frac{169}{13} = 13$.

***