Chapter 7

Area Under Curves

Contents

Ramanujan

In earlier method, we have determined the area of a closed region of the plane, when the region is bounded by line segments. However, if the region is bounded, either partially or wholly, by curves, such a computation cannot be performed by earlier methods. Therefore there is a need for a strong mathematical technique for solving such problems. This comes out to be possible by using the concept of the definite integral.

The definite integral is used to solve many interesting types of problems from various disciplines like economics, finance and probability. The area under certain curves used to solve probability problems.

Area Under Curves

7.1 Introduction

We know the methods of evaluating definite integrals. These integrals are used in evaluating certain types of bounded regions. For evaluation of bounded regions defined by given functions, we shall also require to draw rough sketch of the given function. The process of drawing rough sketch of a given function is called **curve sketching.**

7.2 Procedure of Curve Sketching

(1) **Symmetry:**

(i) Symmetry about x-axis: If all powers of y in equation of the given curve are even, then it is symmetric about x-axis *i.e.,* the shape of the curve above x-axis is exactly identical to its shape below x-axis.

For example, $y^2 = 4ax$ is symmetric about x-axis.

(ii) Symmetry about y-axis: If all power of x in the equation of the given curve are even, then it is symmetric about y-axis

For example, $x^2 = 4ay$ is symmetric about y-axis.

(iii) Symmetry in opposite quadrants or symmetry about origin: If by putting $-x$ for x and $-y$ for y, the equation of a curve remains same, then it is symmetric in opposite quadrants.

For example, $x^2 + y^2 = a^2$ and $xy = a^2$ are symmetric in opposite quadrants.

(iv) Symmetry about the line $y = x$: If the equation of a given curve remains unaltered by interchanging x and y then it is symmetric about the line $y = x$ which passes through the origin and makes an angle of 45^0 with the positive direction of x-axis.

(2) **Origin:** If the equation of curve contains no constant terms then it passes through the origin. Find whether the curve passes through the origin or not.

For examples, $x^2 + y^2 + 4ax = 0$ passes through origin.

(3) **Points of intersection with the axes:** If we get real values of x on putting $y = 0$ in the equation of the curve, then real values of x and $y = 0$ give those points where the curve cuts the x-axis. Similarly by putting $x = 0$, we can get the points of intersection of the curve and y-axis.

For example, the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect the axes at points $(\pm a, 0)$ and $(0, \pm b)$.

(4) **Special points:** Find the points at which $\frac{dy}{dx}=0$, at these points the tangent to the curve is parallel to x-axis. Find the points at which $\frac{dx}{dy}=0$. At these points the tangent to the curve is parallel to y-axis.

(5) **Region:** Write the given equation as $y=f(x)$, and find minimum and maximum values of x which determine the region of the curve.

For example for the curve $xy^2=a^2(a-x) \Rightarrow y=a\sqrt{\frac{a-x}{x}}$

Now y is real, if $0 \le x \le a$, So its region lies between the lines $x = 0$ and $x = a$

(6) **Regions where the curve does not exist:** Determine the regions in which the curve does not exists. For this, find the value of y in terms of x from the equation of the curve and find the value of x for which y is imaginary. Similarly find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exist for these values of x and y.

For example, the values of y obtained from $y^2=4ax$ are imaginary for negative value of x, so the curve does not exist on the left side of y-axis. Similarly the curve $a^2y^2=x^2(a-x)$ does not exist for $x > a$ as the values of y are imaginary for $x > a$.

7.3 Sketching of Some Common Curves

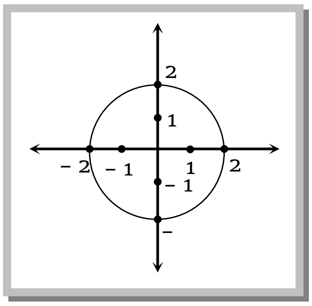
(1) **Straight line:** The general equation of a straight line is $ax+by+c=0$. To draw a straight line, find the points where it meets with the coordinate axes by putting $y = 0$ and $x = 0$ respectively in its equation. By joining these two points, we get the sketch of the line.

(2) **Region represented by a linear inequality:** To find the region represented by linear inequalities $ax+by \le c$ and $ax+by \ge c$, we proceed as follows.

(i) Convert the inequality into equality to obtain a linear equation in x, y.

(ii) Draw the straight line represented by it.

(iii) The straight line obtained in (ii) divides the xy-plane in two parts. To determine the region represented by the inequality choose some convenient points, *e.g.* origin or some point on the coordinate axes. If the coordinates of a point satisfy the inequality, then region containing the point is the required region, otherwise the region not containing the point is the required region.

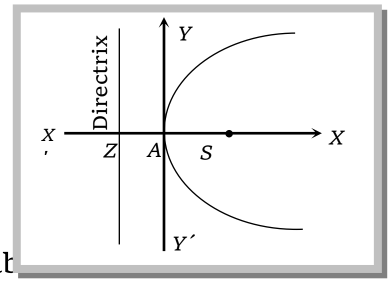


(3) **Circle:** The equation of a circle having centre at (0,0) and radius r is given by $x^2+y^2=r^2$. The equation of a circle having centre at *(h, k)*

and radius r is given by $(x-h)^2+(y-k)^2=r^2$. The general equation of a circle is $x^2+y^2+2gx+2fy+c=0$. This represents the circle whose centre is at (-g,-f) and radius equal to $\sqrt{g^2+f^2-c}$.

The figure of the circle $x^2+y^2=(2)^2$ is given. Here centre is (0,0) and radius is 2.

(4) **Parabola:** There are four standard forms of parabola with vertex at origin and the axis along either of coordinate axis.

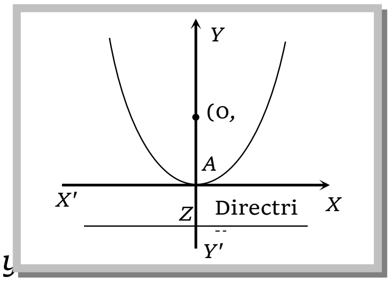


(i) $y^2=\pm 4ax$: For this parabola

(a) Vertex: (0,0) (b) Focus: $(\pm a,0)$

(c) Directrix: $x\pm a=0$ (d) Latus rectum: $4a$

(e) Axis $y=0$ (f) Symmetry : It is symmetric ab

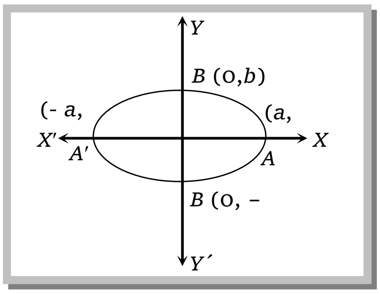


(ii) $x^2=\pm 4ay$: For this parabola

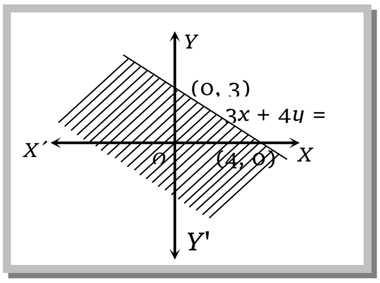
(a) Vertex: (0,0) (b) Focus: $(0,\pm a)$

(c) Directrix: $y\pm a=0$ (d) Latus rectum: $4a$

(e) Axis x = 0 (f) Symmetry: It is symmetric about y

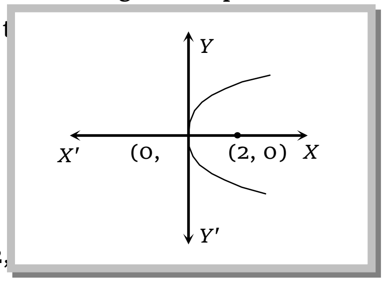


(5) **Ellipse:** The standard equation of the ellipse having its centre at the origin and major and minor axes along the coordinate axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Here $a>b$. The figure of the ellipse is given.



Example: 1 Sketch the region bounded by $3x+4y\le 12$

Solution: Converting the inequality into equation we get $3x+4y=12$. This line meets x-axis at (4,0) and y-axis at (0,3). Joining these two points we obtain the straight line represented by $3x+4y=12$. This straight line divides the plane in two parts. One part contains the origin the other does not contain the origin. Clearly, (0,0) satisfies the inequality $3x+4y\le 12$. So, the region represented by $3x+4y\le 12$ is the region containing t

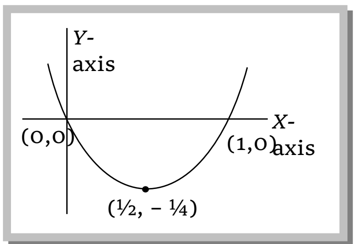


Example: 2 Sketch the parabola $y^2=8x$.

Solution: $y^2=8x$

$y^2=4(2)x$

Here vertex is (0,0) and focus is (2,



Example: 3 Sketch the graph for $y = x^2 - x$.

Solution: We note the following points about the curve.

(i) The curve does not have any kind of symmetry.

(ii) The curve passes through the origin and the tangent at the origin is obtained by equating the lowest degree term to zero.

The lowest degree term is $x + y$. Equation it to zero, we get $x + y = 0$ as the equation of tangent at the origin.

(iii) Putting y=0 in the equation of curve, we get $x^2 - x = 0 \Rightarrow x = 0, 1$. So, the curve crosses x-axis at (0,0) and (1,0).

Putting x=0 in the equation of the curve, we obtain y=0. So, the curve meets y-axis at (0,0) only.

(iv) $y = x^2 - x \Rightarrow \dfrac{dy}{dx} = 2x - 1$ and $\dfrac{d^2y}{dx^2} = 2$

Now, $\dfrac{dy}{dx} = 0 \Rightarrow x = \dfrac{1}{2}$,

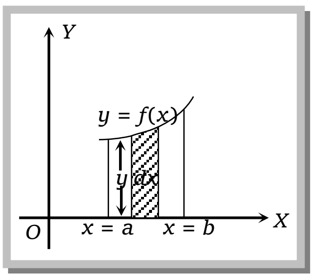
At $x = \dfrac{1}{2}$, $\dfrac{d^2y}{dx^2} > 0$, So $x = \dfrac{1}{2}$ is point of local minima.

(v) $\dfrac{dy}{dx} > 0 \Rightarrow 2x - 1 > 0 \Rightarrow x > \dfrac{1}{2}$, So the curve increases for all $x > \dfrac{1}{2}$ and decreases for all $x < \dfrac{1}{2}$

7.4 Area of Bounded Regions

(1) The area bounded by a cartesian curve $y = f(x)$, x-axis and ordinates $x = a$ and $x = b$ is given by

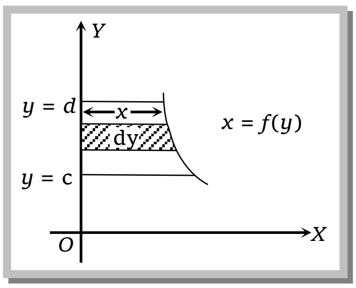
$$\text{Area} = \int_a^b y\,dx = \int_a^b f(x)dx$$



(2) If the curve $y = f(x)$ lies below x-axis, then the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is negative. So, area is given by $\left|\int_a^b y\,dx\right|$

(3) The area bounded by a cartesian curve $x = f(y)$, y-axis and abscissa $y = c$ and $y = d$ is given by

$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)dy$$



(4) If the equation of a curve is in parametric form, say $x = f(t)$, $y = g(t)$ then the area $= \int_a^b y\,dx = \int_{t_1}^{t_2} g(t)\,f'(t)\,dt$ where t_1 and t_2 are the values of t respectively corresponding to the values of a and b of x.

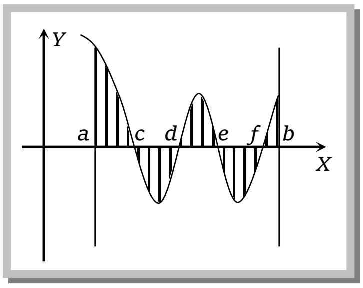
7.5 Sign convention for finding the Areas using Integration

While applying the discussed sign convention, we will discuss the three cases.

Case I: In the expression $\int_a^b f(x)dx$ if $b > a$ and $f(x) > 0$ for all $a \le x \le b$, then this integration will give the area enclosed between the curve $f(x)$, x-axis and the line $x = a$ and $x = b$ which is positive. No need of any modification.

Case II: If in the expression $\int_a^b f(x)dx$ if $b > a$ and $f(x) < 0$ for all $a \le x \le b$, then this integration will calculate to be negative. But the numerical or the absolute value is to be taken to mean the area enclosed between the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$.

Case III. If in the expression $\int_a^b f(x)dx$ where $b > a$ but $f(x)$ changes its sign a numbers of times in the interval $a \le x \le b$, then we must divide the region [a, b] in such a way that we clearly get the points lying between [a, b] where $f(x)$ changes its sign. For the region where $f(x)>0$ we just integrate to get the area in that region and then add the absolute value of the integration calculated in the region where $f(x)<0$ to get the desired area between the curve $y = f(x)$, x-axis and the line $x = a$ and $x = b$.



Hence, if $f(x)$ is as in above figure, the area enclosed by $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ is given by

$$A = \int_a^c f(x)dx + \left|\int_c^d f(x)dx\right| + \int_d^e f(x)dx + \left|\int_e^f f(x)dx\right| + \int_f^b f(x)dx$$

Example: 4 The area (in square units) enclosed by the curve $x^2y = 36$, the x-axis and the lines $x = 6$ and $x = 9$ is

[Kerala (Engg.) 2000]

(a) 2 (b) 1 (c) 4 (d) 3

Solution: (a) Required area $= \int_6^9 y\,dx = \int_6^9 \frac{36}{x^2}dx$ [Given $x^2y = 36 \Rightarrow y = \frac{36}{x^2}$]

$$= \left[-\frac{36}{x}\right]_6^9 = -\left[\frac{36}{9} - \frac{36}{6}\right] = -[4-6] = 2\,.$$

Example: 5 The area bounded by the x-axis, the curve $y = f(x)$ and the lines $x = 1$, $x = b$ is equal to $\sqrt{b^2+1}-\sqrt{2}$ for all $b > 1$, then $f(x)$ is **[MP PET 2000]**

(a) $\sqrt{x-1}$ (b) $\sqrt{x+1}$ (c) $\sqrt{x^2-1}$ (d) $\dfrac{x}{\sqrt{1+x^2}}$

Solution: (d) $\int_1^b f(x)\,dx = \sqrt{b^2+1}-\sqrt{2} = \left[\sqrt{x^2+1}\right]_1^b$

$\Rightarrow f(x) = \dfrac{1}{2}.\dfrac{2x}{\sqrt{x^2+1}}$

Hence $f(x) = \dfrac{x}{\sqrt{1+x^2}}$.

Example: 6 The area of the region bounded by the curve $y = x - x^2$ between $x = 0$ and $x = 1$ is **[Pb. CET 1994, 89]**

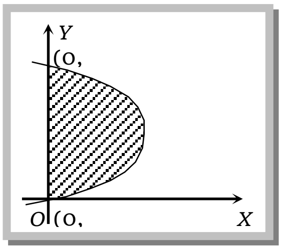
(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{3}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{5}{6}$

Solution: (a) Required Area $= \int_0^1 (x-x^2)dx = \left[\dfrac{x^2}{2}-\dfrac{x^3}{3}\right]_0^1 = \dfrac{1}{2}-\dfrac{1}{3} = \dfrac{1}{6}$.

Example: 7 Find the area bounded between the curve $y^2 = 2y - x$ and y-axis .

(a) $\dfrac{4}{3}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{1}{3}$ (d) 5

Solution: (a) The area between the given curve $x = 2y - y^2$ and y- axis will be as shown



$\therefore$ Required Area $= \int_0^2 (2y - y^2)dy$

$= \left[y^2 - \dfrac{y^3}{3}\right]_0^2 = \dfrac{4}{3}$

Example: 8 Find the area bounded by the curves $x = a \cos t$, $y = b \sin t$ in the first quadrant

(a) $\dfrac{\pi ab}{4}$ (b) $\dfrac{\pi a^2 b}{4}$ (c) $\dfrac{\pi ab^2}{4}$ (d) None of these

Solution: (a) Clearly the given equation are the parametric equation of ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Curve meet the x-axis in the first quadrant at $(a,0)$

$\therefore$ Required area $= \int_o^a y\,dx = \int_{\frac{\pi}{2}}^0 (b \sin t)(-a \cos t)dt = ab\int_0^{\pi/2} \sin^2 t\,dt = \left(\dfrac{\pi ab}{4}\right)$

$(\because \text{At } x = 0, t = \pi/2 \text{ and } x = a, t = 0)$

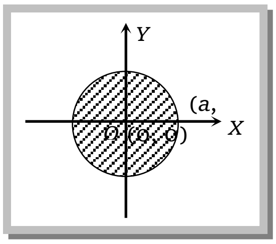
7.6 Symmetrical Area

If the curve is symmetrical about a coordinate axis (or a line or origin), then we find the area of one symmetrical portion and multiply it by the number of symmetrical portions to get the required area.

Example: 9 Find the whole area of circle $x^2+y^2=a^2$

(a) π (b) πa^2 (c) πa^3 (d) a^2

Solution: (b) The required area is symmetric about both the axis as shown in figure



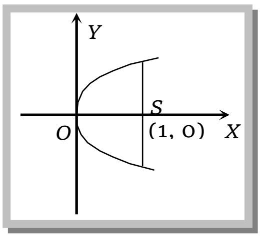
$\therefore$ Required area = $4\int_0^a \sqrt{a^2-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= 4\left[\frac{\pi}{2}\times\frac{a^2}{2}\right] = \pi a^2$

Example: 10 Find the area bounded by the parabola $y^2=4x$ and its latus rectum **[Rajasthan PET 1997, 94, 92, 84]**

(a) $\frac{8}{3}$ (b) $\frac{4}{3}$ (c) $\frac{16}{3}$ (d) None of these

Solution: (a) Since the curve is symmetrical about x-axis, therefore the required area



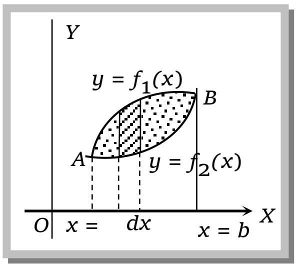
$= 2\int_0^1 y\,dx = 2\int_0^1 \sqrt{4x}\,dx$

$= 4.\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 = \frac{8}{3}$

7.7 Area between Two curves

(1) When both curves intersect at two points and their common area lies between these points:

If the curves $y_1=f_1(x)$ and $y_2=f_2(x)$, where $f_1(x)>f_2(x)$ intersect in two points $A(x=a)$ and $B(x=b)$, then common area between the curves is $=\int_a^b (y_1-y_2)dx$

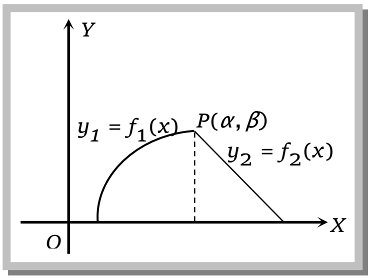


$= \int_a^b [f_1(x)-f_2(x)]dx$

(2) When two curves intersect at a point and the area between them is bounded by x-axis:

Area bounded by the curves $y = f_1(x), y_2 = f_2(x)$ and $x-$axis is

$$= \int_a^{\alpha} f_1(x)dx + \int_{\alpha}^{b} f_2(x)dx$$



Where $P(\alpha,\beta)$ is the point of intersection of the two curves.

(3) **Positive and negative area :** Area is always taken as positive. If some part of the area lies above the x-axis and some part lies below x-axis, then the area of two parts should be calculated separately and then add their numerical values to get the desired area.

Important Tips

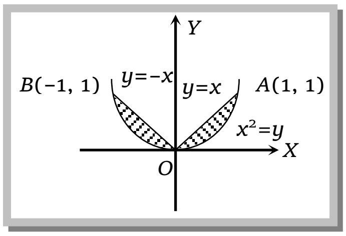
- *The area of the region bounded by* $y^2 = 4ax$ *and* $x^2 = 4by$ *is* $\frac{16ab}{3}$ *square units.*
- *The area of the region bounded by* $y^2 = 4ax$ *and* $y = mx$ *is* $\frac{8a^2}{3m^3}$ *square units*
- *The area of the region bounded by* $y^2 = 4ax$ *and its latus rectum is* $\frac{8a^2}{3}$ *square units*
- *The area of the region bounded by one arch of sin (ax) or cos (ax) and x-axis is* $\frac{2}{a}$ *sq. units*
- *Area of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is* πab *sq. units.*
- *Area of region bounded by the curve y = sin x , x-axis and the line x =0 and x=* 2π *is 4 unit.*

Example: 11 The area of the region bounded by the curves $y = x^2$ and $y = |x|$ is **[Roorkee (Qualifying) 2000]**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{5}{6}$ (d) $\frac{5}{3}$

Solution: (b) Required area $= 2\left[\int_0^1 x\,dx - \int_0^1 x^2 dx\right] = 2\left[\left\{\frac{x^2}{2}\right\}_0^1 - \left\{\frac{x^3}{3}\right\}_0^1\right]$

$= 2\left[\left(\frac{1}{2}-0\right)-\left(\frac{1}{3}-0\right)\right] = 2\left[\frac{1}{2}-\frac{1}{3}\right] = 2\left[\frac{1}{6}\right] = \frac{1}{3}$.



Example: 12 The area (in square units) bounded by the curve $y = x^3$, $y = x^2$ and the ordinates $x = 1$, $x = 2$ is **[EAMCET 2000**

(a) $\frac{17}{12}$ (b) $\frac{12}{17}$ (c) $\frac{2}{7}$ (d) $\frac{7}{2}$

Solution: (a) Required area $= \int_1^2 (x^3 - x^2)dx = \left[\frac{x^4}{4}-\frac{x^3}{3}\right]_1^2 = \left(4-\frac{8}{3}\right)-\left(\frac{1}{4}-\frac{1}{3}\right) = \frac{4}{3}+\frac{1}{12} = \frac{16+1}{12} = \frac{17}{12}$.

Example: 13 The area of the region bounded by the curve $y=2x-x^2$ and line $y=x$ is **[Pb. CET 2000; Roorkee 1992]**

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

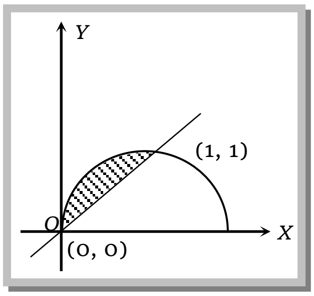
Solution: (d) The given curve is $y = 2x - x^2$

$\Rightarrow y = -(x^2 - 2x + 1) + 1$

$\Rightarrow y - 1 = -(x-1)^2$, it represents a downward parabola with vertex (1,1)

Its points of intersection with the line $y = x$ are (0,0) and (1,1).

Required area = shaded region

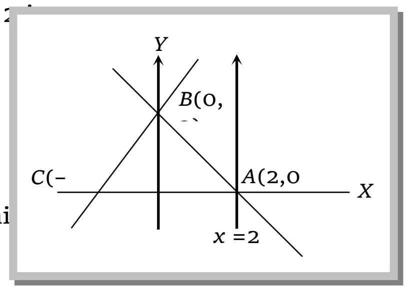


$$= \int_0^1 (2x - x^2)dx - \int_0^1 x\,dx = \int_0^1 (x - x^2)dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Example: 14 Area bounded by the lines $y = 2 + x$, $y = 2 - x$ and $x = 2$ is

(a) 3 (b) 4

(c) 8 (d) 16



Solution: (b) Given lines are $y = x + 2$, $y = -x + 2$, $x = 2$

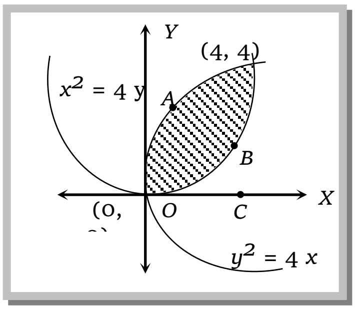
Hence required area = Area of $\Delta CAB = \frac{1}{2}(2)(4) = 4$ sq. unit

Example: 15 The area bounded by the curve $y^2 = 4x$ and $x^2 = 4y$ is

[Karnataka CET 1999,2003; MP PET 1997; SCRA 1986; Rajasthan PET 1988, 99,97]

(a) $\frac{16}{3}$ sq. units (b) $\frac{3}{16}$ sq. units (c) $\frac{14}{3}$ sq. units (d) $\frac{3}{14}$ sq. units

Solution: (a) Required area $= \int_0^4 (OABC - ODBC)$ Region $= \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right)dx = \frac{16}{3}$ square unit.



Trick : From Important Tips' the area of the region bounded by $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16ab}{3}$ square unit.

Here $y^2 = 4x$ and $x^2 = 4y$, so $a = 1$ and $b = 1$

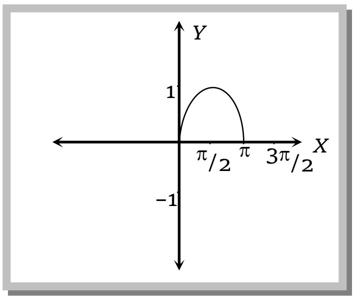
Required area = $\frac{16}{3}(1)\ (1) = \frac{16}{3}$ square unit.

Example: 16 The area of the bounded region by the curve $y = \sin x$, the x-axis and the line $x = 0$ and $x = \pi$ is

[Rajasthan PET 1989, 92]

(a) 4 (b) 2 (c) 0 (d) None of these

Solution: (b) Required area = $\int_0^{\pi} \sin x\,dx$



$= 2\int_0^{\pi/2} \sin x\,dx = 2[-\cos x]_0^{\pi/2} = 2\ [\ (-\cos \pi/2) - (-\cos 0)\ ] = 2(1)$

= 2 square unit.

Trick : For the curve $y = \sin x$ or $\cos x$, the area of

$\int_0^{\pi/2} \sin x\,dx = 1, \int_0^{\pi} \sin x\,dx = 2, \int_0^{3\pi/2} \sin x\,dx = 3, \int_0^{2\pi} \sin x\,dx = 4$ and so on.

Example: 17 The area enclosed by the parabola $y^2 = 8x$ and the line $y = 2x$ is

(a) $\frac{4}{3}$ (b) $\frac{3}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

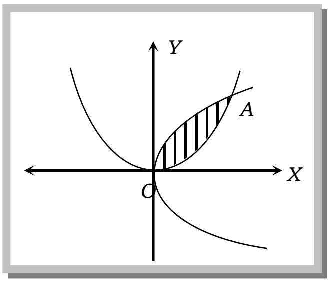
Solution: (a) Solve the equation $y^2 = 8x$ and the line $y = 2x$, we get the point of intersection. Then find the required area bounded by this region. It is $\frac{4}{3}$.

Trick : Required area $=\frac{8(2)^2}{3(2)^3}=\frac{32}{24}=\frac{4}{3}$ [$\because$ Area bounded by $y^2=4ax$ and $y=mx$ is $\frac{8a^2}{3m^3}$. Here $a=2, m=2$]

Example: 18 If the area bounded by $y=ax^2$ and $x=ay^2$, $a>0$, is 1, then a =

(a) 1 (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{\sqrt{3}}$

Solution: (b) The x coordinate of A is $\frac{1}{a}$



According to the given condition

$$1=\int_0^{\frac{1}{a}}\left(\sqrt{\frac{x}{a}}-ax^2\right)dx=\frac{1}{\sqrt{a}}.\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{\frac{1}{a}}-\frac{a}{3}[x^3]_0^{1/a}$$

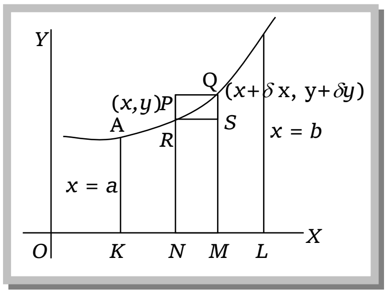
$$\Rightarrow a=\frac{1}{\sqrt{3}}$$

7.8 Volumes and Surfaces of Solids of Revolution

If a plane curve is revolved about some axis in the plane of the curve, then the body so generated is known as solid of revolution. The surface generated by the perimeter of the curve is known as surface of revolution and the volume generated by the area is called volume of revolution.

For example, a right angled triangle when revolved about one of its sides (forming the right angle) generates a right circular cones.

(1) **Volumes of solids of revolution:**



(i) The volume of the solid generated by the revolution, about the x-axis, of the area bounded by the curve $y = f(x)$, the ordinates at $x = a$, $x = b$ and the x-axis is equal to $\pi\int_a^b y^2\,dx$.

(ii) The revolution of the area lying between the curve $x = f(y)$, the y-axis and the lines $y=a$ and $y=b$ is given by (interchanging x and y in the above formulae) $\int_a^b \pi x^2\,dy$.

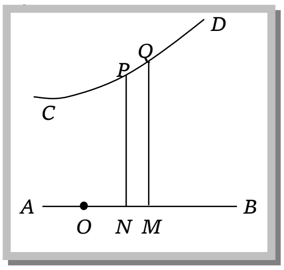
(iii) If the equation of the generating curve be given by $x=f_1(t)$ and $y=f_2(t)$ and it is revolved about x-axis, then the formula corresponding to $\int_a^b \pi y^2\,dx$ becomes $\int_{t_1}^{t_2}\pi\{f_2(t)\}^2\,d\{f_1(t)\}$, where f_1 and f_2 are the values of t corresponding to $x = a$ and $x = b$

(iv) If the curve is given by an equation in polar co-ordinates, say $r=f(\theta)$, and the curve revolves about the initial line, the volume generated

$=\pi\int_a^b y^2\,dx=\pi\int_\alpha^\beta y^2\left(\frac{dx}{d\theta}\right).d\theta$, where α and β are the values of θ corresponding to $x = a$ and $x = b$

Now $x = r\cos\theta,\ y = r\sin\theta$. Hence the volume $= \pi\int_{\alpha}^{\beta} r^2 \sin^2\theta \frac{d}{d\theta}(r\cos\theta)d\theta$

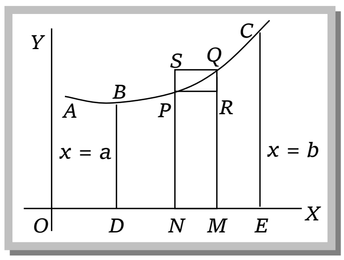
(v) If the generating curve revolves about any line AB (which is different from either of the axes), then the volume of revolution is



Note : ❑ The volume of the solid generated by revolving the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$ about the initial line is $\frac{2}{3}\pi\int_{\alpha}^{\beta} r^3 \sin\theta\, d\theta$.

❑ The volume in the case when the above area is revolved about the line $\theta = \frac{\pi}{2}$ is $\frac{2}{3}\pi \int_{\alpha}^{\beta} r^3 \cos\theta\, d\theta.$

(2) **Area of surfaces of revolution:**



(i) The curved surface of the solid generated by the revolution, about the x- axis, of the area bounded by the curve $y = f(x)$, the ordinates at $x = a$, $x = b$ and the x-axis is equal to $2\pi\int_{x=a}^{x=b} y\, ds$.

(ii) If the arc of the curve $y = f(x)$ revolves about y-axis, then the area of the surface of revolution (between proper limits) $= 2\pi\int x\, ds,$ where $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

(iii) If the equation of the curve is given in the parametric form $x = f_1(t)$ and $y = f_2(t)$, and the curve revolves about x-axis, then we get the area of the surface of revolution $= 2\pi\int_{t=t_1}^{t=t_2} y ds = 2\pi\int_{t=t_1}^{t=t_2} f_2(t) ds$

$= 2\pi\int_{t_1}^{t_2} f_2(t)\sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}dt$, where t_1 and t_2 are the values of the parameter t corresponding to $x = a$ and $x = b$.

(iv) If the equation of the curve is given in polar form then the area of the surface of revolution about x-axis $= 2\pi\int y\, ds = 2\pi\int (r\sin\theta)\frac{dS}{d\theta}.d\theta = 2\pi\int r\sin\theta.\sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}d\theta$ between proper limits.

Example: 19 The part of circle $x^2+y^2=9$ in between $y=0$ and $y=2$ is revolved about y-axis. The volume of generating solid will be

[UPSEAT 1999]

(a) $\frac{46}{3}\pi$ (b) 12π (c) 16π (d) 28π

Solution: (a) The part of circle $x^2+y^2=9$ in between $y=0$ and $y=2$ is revolved about y- axis. Then a frustum of sphere will be formed.

The volume of this frustum $=\pi\int_0^2 x^2 dy = \pi\int_0^2 (9-y^2)dy$

$$=\pi\left[9y-\frac{1}{3}y^3\right]_0^2 = \pi\left[9\times 2-\frac{1}{3}(2)^3-(9.0-\frac{1}{3}.0)\right] = \frac{46}{3}\pi \text{ cubic unit.}$$

Example: 20 The part of straight line $y=x+1$ between $x=2$ and $x=3$ is revolved about x-axis, then the curved surface of the solid thus generated is
[UPSEAT 2000]

(a) $\frac{37\pi}{3}$ (b) $\frac{7\pi}{\sqrt{2}}$ (c) 37π (d) $7\pi\sqrt{2}$

Solution: (d) Curved surface $=\int_a^b 2\pi y\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$. Given that $a=2$, $b=3$ and $y=x+1$ on differentiating with respect to x

$\frac{dy}{dx}=1+0$ or $\frac{dy}{dx}=1$. Therefore, curved surface $=\int_2^3 2\pi(x+1)\sqrt{[1+(1)^2]}\,dx$

$$=\int_2^3 2\pi(x+1)\sqrt{2}\,dx = 2\sqrt{2}\pi\int_2^3(x+1)dx = 2\sqrt{2}\pi\left[\frac{(x+1)^2}{2}\right]_2^3 = \frac{2\sqrt{2}}{2}\pi[(3+1)^2-(2+1)^2]$$

$$=\sqrt{2}\pi(16-9)=7\sqrt{2}\pi=7\pi\sqrt{2}$$