Chapter 6

Definite Integral

Contents

J. Bernoulli

Newton introduced the basic notion of inverse function called the anti-derivative (integral) or the inverse method of tangents.

During 1684-86 A.D. **Leibnitz** *published an article in the Acta Eruditorum which he called calculas summatorius, since it was connected with the summation of a number of infinitely small areas, whose sum, he indicated by the symbol* $\int$. *In* 1696 A.D. *he followed a suggestion made by* **J.Bernoulli** *and changed this article to calculus integrali. This corresponded to Newton's inverse method of tangents.*

Both **Newton** *and* **Leibnitz** *adopted quite independent lines of approach which were radically different. However, respective theories accomplished results that were practically identical. Leibnitz used the notion of definite integral and what is quite certain is that he first clearly appreciated tie up between the anti-derivative and the definite integral. The discovery that differentiation and integration are inverse operations belongs to Newton and Leibnitz.*

Definite Integral

6.1 Definition

Let $\phi(x)$ be the primitive or anti-derivative of a function $f(x)$ defined on $[a, b]$ *i.e.,* $\frac{d}{dx}[\phi(x)] = f(x)$. Then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)dx$ and is defined as $[\phi(b) - \phi(a)]$ *i.e.,* $\int_a^b f(x)dx = \phi(b) - \phi(a)$. This is also called Newton Leibnitz formula.

The numbers a and b are called the limits of integration, 'a' is called the lower limit and 'b' the upper limit. The interval $[a, b]$ is called the interval of integration. The interval $[a, b]$ is also known as range of integration.

Important Tips

☞ *In the above definition it does not matter which anti-derivative is used to evaluate the definite integral, because if* $\int f(x)dx = \phi(x) + c$, *then* $\int_a^b f(x)dx = [\phi(x) + c]_a^b = (\phi(b) + c) - (\phi(a) + c) = \phi(b) - \phi(a)$.

In other words, to evaluate the definite integral there is no need to keep the constant of integration.

☞ *Every definite integral has a unique value.*

Example: 1 $\int_{-1}^{3}\left[\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right]dx =$ **[Karnataka CET 2000]**

(a) π (b) 2π (c) 3π (d) None of these

Solution: (b) $I = \int_{-1}^{3}\left[\tan^{-1}\frac{x}{x^2+1} + \cot^{-1}\frac{x}{x^2+1}\right]dx$

$\Rightarrow \quad I = \int_{-1}^{3}\frac{\pi}{2}dx \Rightarrow I = \frac{\pi}{2}[x]_{-1}^{3} = \frac{\pi}{2}[3+1] = 2\pi$.

Example: 2 $\int_0^{\pi}\sin^2 x\, dx$ is equal to **[MP PET 1999]**

(a) π (b) $\pi/2$ (c) 0 (d) None of these

Solution: (b) $I = \frac{1}{2}\int_0^{\pi} 2\sin^2 x\, dx = \frac{1}{2}\int_0^{\pi}[1 - \cos 2x]dx$

$\Rightarrow I = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} \Rightarrow I = \frac{1}{2}[\pi] = \frac{\pi}{2}$.

6.2 Definite Integral as the Limit of a Sum

Let $f(x)$ be a single valued continuous function defined in the interval $a \le x \le b$, where a and b are both finite. Let this interval be divided into n equal sub-intervals, each of width h by inserting (n – 1) points $a + h, a + 2h, a + 3h......a + (n-1)h$ between a and b. Then $nh = b - a$.

Now, we form the sum $hf(a)+hf(a+h)+hf(a+2h)+........+hf(a+rh)+......+hf[a+(n-1)h]$

$$= h[f(a)+f(a+h)+f(a+2h)+.....+f(a+rh)+....+f\{a+(n-1)h\}]$$

$$= h\sum_{r=0}^{n-1} f(a+rh)$$

where, $a+nh=b \Rightarrow nh=b-a$

The $\lim\limits_{h\to 0} h\sum\limits_{r=0}^{n-1} f(a+rh)$, if it exists is called the **definite integral** of $f(x)$ with respect to x between the limits a and b and we denote it by the symbol $\int_a^b f(x)dx$.

Thus, $\int_a^b f(x)dx = \lim\limits_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+......+f\{a+(n-1)h\}] \Rightarrow \int_a^b f(x)dx = \lim\limits_{h\to 0} h\sum\limits_{r=0}^{n-1} f(a+rh)$

where, $nh=b-a$, a and b being the limits of integration.

The process of evaluating a definite integral by using the above definition is called integration from the first principle or integration as the limit of a sum.

Important Tips

☞ *In finding the above sum, we have taken the left end points of the subintervals. We can take the right end points of the sub-intervals throughout.*

Then we have, $\int_a^b f(x)dx = \lim\limits_{h\to 0} h[f(a+h)+f(a+2h)+.....+f(a+nh)]$, $\Rightarrow \int_a^b f(x)dx = h\sum\limits_{r=1}^{n} f(a+rh)$

where, $nh=b-a$.

☞ $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}\,(\beta>\alpha)=\pi$

☞ $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\;dx=\frac{\pi}{8}(\beta-\alpha)^2$

☞ $\int_a^b \sqrt{\frac{x-a}{b-x}}\;dx=\frac{\pi}{2}(b-a)$

☞ $\int_a^b f(x)\;dx=\frac{1}{n}\int_{na}^{nb} f(x)\;dx$

☞ $\int_{a-c}^{b-c} f(x+c)\;dx=\int_a^b f(x)\;dx$ or $\int_{a+c}^{b+c} f(x-c)\;dx=\int_a^b f(x)\;dx$

Some useful results for evaluation of definite integrals as limit for sums

(i) $1+2+3+.......+n=\frac{n(n+1)}{2}$

(ii) $1^2+2^2+3^2+........+n^2=\frac{n(n+1)(2n+1)}{6}$

(iii) $1^3+2^3+3^3+.......+n^3=\left[\frac{n(n+1)}{2}\right]^2$

(iv) $a+ar+ar^2+........+ar^{n-1}=\frac{a(r^n-1)}{r-1}, r\neq 1,\ r>1$

(v) $a+ar+ar^2+..............+ar^{n-1}=\frac{a(1-r^n)}{1-r},\ r\neq 1,\ r<1$

(vi) $\sin a+\sin(a+h)+........+\sin[a+(n-1)h]=\sum\limits_{r=0}^{n-1}[\sin(a+nh)]=\dfrac{\sin\left\{a+\left(\frac{n-1}{2}\right)h\right\}\sin\left\{\frac{nh}{2}\right\}}{\sin\left(\frac{h}{2}\right)}$

(vii) $\cos a + \cos(a+h) + \cos(a+2h) + \ldots.. + \cos[a+(n-1)h] = \sum_{r=0}^{n-1}[\cos(a+nh)] = \dfrac{\cos\left\{a+\left(\frac{n-1}{2}\right)h\right\}\sin\left\{\frac{nh}{2}\right\}}{\sin\left(\frac{h}{2}\right)}$

(viii) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\ldots\ldots\ldots\ldots\ldots\ldots\infty = \frac{\pi^2}{12}$ (ix) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}\ldots\ldots\ldots\ldots\ldots\infty = \frac{\pi^2}{6}$

(x) $1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\ldots\ldots\infty = \frac{\pi^2}{8}$ (xi) $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\ldots\ldots\ldots\ldots\ldots\infty = \frac{\pi^2}{24}$

(xii) $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$ (xiii) $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$ and $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$

6.3 Evaluation of Definite Integral by Substitution

When the variable in a definite integral is changed, the substitutions in terms of new variable should be effected at three places.

(i) In the integrand (ii) In the differential say, dx (iii) In the limits

For example, if we put $\phi(x) = t$ in the integral $\int_a^b f\{\phi(x)\}\phi'(x)dx$, then $\int_a^b f\{\phi(x)\}\phi'(x)dx = \int_{\phi(a)}^{\phi(b)} f(t)\,dt$.

Important Tips

☞ $\int_0^{\pi} \frac{dx}{1+\sin x} = 2$

☞ $\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \sqrt{2}\,\log\left(\sqrt{2}+1\right)$

☞ $\int_0^{\pi/2} \log(\tan x)\,dx = 0$

☞ $\int_0^{a} \frac{dx}{1+e^{f(x)}} = \frac{a}{2}$, where $f(a-x) = -f(x)$

☞ $\int_0^{a} \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{2}$

☞ $\int_0^{a} \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$

☞ $\int_0^{a} \sqrt{a^2-x^2}\,dx = \frac{\pi a^2}{4}$

Example: 3 If $h(a) = h(b)$, then $\int_a^b [f(g[h(x)])]^{-1} f'(g[h(x)])g'[h(x)]h'(x)dx$ is equal to **[MP PET 2001]**

(a) 0 (b) $f(a) - f(b)$ (c) $f[g(a)] - f[g(b)]$ (d) None of these

Solution: (a) Put $f(g[h(x)]) = t \Rightarrow f'(g[h(x)])g'[h(x)]h'(x)dx = dt$

$\therefore \int_{f(g[h(a)])}^{f(g[h(b)])} t^{-1}dt = [\log(t)]_{f(g[h(a)])}^{f(g[h(b)])} = 0$ $[\because h(a) = h(b)]$

Example: 4 The value of the integral $\int_{e^{-1}}^{e^2}\left|\frac{\log_e x}{x}\right|dx$ is **[IIT 2000]**

(a) 3/2 (b) 5/2 (c) 3 (d) 5

Solution: (b) Put $\log_e x = t \Rightarrow e^t = x$

$\therefore dx = e^t dt$

and limits are adjusted as –1 to 2

$\therefore I = \int_{-1}^{2}\left|\frac{t}{e^t}\right|e^t dt = \int_{-1}^{2}|t|\,dt \Rightarrow I = \int_{-1}^{0} -t\,dt + \int_0^2 t\,dt \Rightarrow I = \left[\frac{-t^2}{2}\right]_{-1}^{0} + \left[\frac{t^2}{2}\right]_0^2 \Rightarrow I = 5/2$

Example: 5 $\int_0^{\pi/2} \frac{dx}{1+\sin x}$ equals **[MNR 1983; Rajasthan PET 1990; Kurukshetra CEE 1997]**

(a) 0 (b) 1 (c) –1 (d) 2

Solution: (b) $I = \int_0^{\pi/2} \frac{dx}{\sin^2 x/2 + \cos^2 x/2 + 2\sin x/2 \cos x/2}$

$$I = \int_0^{\pi/2} \frac{dx}{(\sin x/2 + \cos x/2)^2} = \int_0^{\pi/2} \frac{\sec^2 x/2}{(1+\tan x/2)^2} dx$$

Put $(1+\tan x/2) = t \Rightarrow \frac{1}{2}\sec^2 x/2\, dx = dt$

$$\therefore\ I = 2\int_1^2 \frac{dt}{t^2} = -2\left[\frac{1}{t}\right]_1^2 = -2\left[\frac{1}{2} - \frac{1}{1}\right] = 1$$

6.4 Properties of Definite Integral

(1) $\int_a^b f(x)dx = \int_a^b f(t)dt$ *i.e.*, The value of a definite integral remains unchanged if its variable is replaced by any other symbol.

Example: 6 $\int_3^6 \frac{1}{x+1} dx$ is equal to

(a) $[\log(x+1)]_3^6$ (b) $[\log(t+1]_3^6$ (c) Both (a) and (b) (d) None of these

Solution: (c) $I = \int_3^6 \frac{1}{x+1} dx = [\log(x+1)]_3^6$, $I = \int_3^6 \frac{1}{t+1} dt = [\log(t+1)]_3^6$

(2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ *i.e.*, by the interchange in the limits of definite integral, the sign of the integral is changed.

Example: 7 Suppose f is such that $f(-x) = -f(x)$ for every real x and $\int_0^1 f(x)\,dx = 5$, then $\int_{-1}^0 f(t)\,dt =$ **[MP PET 2000]**

(a) 10 (b) 5 (c) 0 (d) –5

Solution: (d) Given, $\int_0^1 f(x)dx = 5$

Put $x = -t \Rightarrow dx = -dt$

$$\therefore\ I = -\int_0^{-1} f(-t)dt = -\int_{-1}^0 f(t)dt \Rightarrow I = -5$$

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, (where $a < c < b$)

or $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + ... + \int_{c_n}^b f(x)dx$; (where $a < c_1 < c_2 < ....c_n < b$)

Generally this property is used when the integrand has two or more rules in the integration interval.

Important Tips

☞ $\int_a^b (|x-a| + |x-b|)\, dx = (b-a)^2$

Note : ❑ Property (3) is useful when $f(x)$ is not continuous in [a, b] because we can break up the integral into several integrals at the points of discontinuity so that the function is continuous in the sub-intervals.

❑ The expression for $f(x)$ changes at the end points of each of the sub-interval. You might at times be puzzled about the end points as limits of integration. You may not be sure for $x = 0$ as the limit of the first integral or the next one. In fact, it is immaterial, as the area of the line is always zero. Therefore, even if you write $\int_{-1}^{0} (1-2x)\,dx,$ whereas 0 is not included in its domain it is deemed to be understood that you are approaching $x = 0$ from the left in the first integral and from right in the second integral. Similarly for $x = 1$.

Example: 8 $\int_{-2}^{2} |1-x^2|\,dx$ is equal to **[IIT 1989; BIT Ranchi 1996; Kurukshetra CEE 1998]**

(a) 2 (b) 4 (c) 6 (d) 8

Solution: (b) $I = \int_{-2}^{2} |1-x^2|\,dx = \int_{-2}^{-1} |1-x^2|\,dx + \int_{-1}^{1} |1-x^2|\,dx + \int_{1}^{2} |1-x^2|\,dx$

$\Rightarrow I = -\int_{-2}^{-1} (1-x^2)dx + \int_{-1}^{1} (1-x^2)dx - \int_{1}^{2} (1-x^2)dx \Rightarrow I = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$.

Example: 9 $\int_{0}^{1.5} [x^2]dx$, where [.] denotes the greatest integer function, equals **[DCE 2000, 2001; IIT 1988; AMU 1998]**

(a) $2+\sqrt{2}$ (b) $2-\sqrt{2}$ (c) $1+\sqrt{2}$ (d) $\sqrt{2}-1$

Solution: (b) $I = \int_{0}^{1.5} [x^2]dx = \int_{0}^{1} [x^2]dx + \int_{1}^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{1.5} [x^2]dx \Rightarrow I = 0 + \int_{1}^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{1.5} 2dx = \sqrt{2} - 1 + 3 - 2\sqrt{2} \Rightarrow I = 2 - \sqrt{2}$

Example: 10 If $f(x) = \begin{cases} e^{\cos x}.\sin x, & |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)dx =$ **[IIT 2000]**

(a) 0 (b) 1 (c) 2 (d) 3

Solution: (c) $|x| \le 2 \Rightarrow -2 \le x \le 2$ and $f(x) = e^{\cos x} \sin x$ is an odd function.

$\therefore I = \int_{-2}^{3} f(x)dx = \int_{-2}^{2} f(x)dx + \int_{2}^{3} f(x)dx \Rightarrow I = 0 + \int_{2}^{3} 2dx = [2x]_2^3 = 2$ $\left[\because \int_{-a}^{a} f(x) = 0 \text{ if } f(x) \text{ is odd and in } (2, 3)\ f(x) \text{ is } 2\right]$

(4) $\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx$: This property can be used only when lower limit is zero. It is generally used for those complicated integrals whose denominators are unchanged when x is replaced by $(a - x)$.

Following integrals can be obtained with the help of above property.

(i) $\int_{0}^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \int_{0}^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x} dx = \frac{\pi}{4}$

(ii) $\int_{0}^{\pi/2} \frac{\tan^n x}{1+\tan^n x} dx = \int_{0}^{\pi/2} \frac{\cot^n x}{1+\cot^n x} dx = \frac{\pi}{4}$

(iii) $\int_{0}^{\pi/2} \frac{1}{1+\tan^n x} dx = \int_{0}^{\pi/2} \frac{1}{1+\cot^n x} dx = \frac{\pi}{4}$

(iv) $\int_{0}^{\pi/2} \frac{\sec^n x}{\sec^n x + \text{cosec}^n x} dx = \int_{0}^{\pi/2} \frac{\text{cosec}^n x}{\text{cosec}^n x + \sec^n x} dx = \frac{\pi}{4}$

(v) $\int_{0}^{\pi/2} f(\sin 2x)\sin x\,dx = \int_{0}^{\pi/2} f(\sin 2x)\cos x\,dx$

(vi) $\int_{0}^{\pi/2} f(\sin x)dx = \int_{0}^{\pi/2} f(\cos x)dx$

(vii) $\int_{0}^{\pi/2} f(\tan x)dx = \int_{0}^{\pi/2} f(\cot x)dx$

(viii) $\int_{0}^{1} f(\log x)dx = \int_{0}^{1} f[\log(1-x)]dx$

(ix) $\int_0^{\pi/2} \log \tan x dx = \int_0^{\pi/2} \log \cot x dx$

(x) $\int_0^{\pi/4} \log(1+\tan x) dx = \frac{\pi}{8}\log 2$

(xi) $\int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \log \cos x dx = \frac{-\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$

(xii) $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{a\sec x + b\,\mathrm{cosec}\, x}{\sec x + \mathrm{cosec}\, x} dx = \int_0^{\pi/2} \frac{a\tan x + b\cot x}{\tan x + \cot x} dx = \frac{\pi}{4}(a+b)$

Example: 11 $\int_0^{\pi} e^{\sin^2 x}\cos^3 x dx =$

OR

For any integer n, $\int_0^{\pi} e^{\sin^2 x}\cos^3(2n+1)x dx =$ **[MP PET 2002; MNR 1992, 98]**

(a) –1 (b) 0 (c) 1 (d) None of these

Solution: (b) Let, $f_1(x) = \cos^3 x = -f(\pi - x)$

and $f_2(x) = \cos^3(2n+1)x = -f(\pi - x)$

$\therefore$ $I = 0$.

Example: 12 $\int_0^{2a} \frac{f(x)}{f(x) + f(2a-x)} dx$ is equal to **[IIT 1988; Kurukshetra CEE 1999; Karnataka CET 2000]**

(a) a (b) $a/2$ (c) $2a$ (d) 0

Solution: (a) $I = \int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)} dx = \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)} dx$

$2I = \int_0^{2a} \frac{f(x)+f(2a-x)}{f(x)+f(2a-x)} dx = \int_0^{2a} dx = [x]_0^{2a} = 2a$

$\therefore I = a$.

Example: 13 $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ is equal to **[MNR 1989, 90; Rajasthan PET 1991, 95]**

(a) $\pi/4$ (b) ∞ (c) –1 (d) 1

Solution: (a) We know, $\int_0^{\pi/2} \frac{\tan^n x dx}{1+\tan^n x} = \frac{\pi}{4}$ for any value of n

$\therefore$ $I = \pi/4$.

(5) $\int_{-a}^{a} f(x) dx = \int_0^a f(x) + f(-x) dx$.

In special case : $\int_{-a}^{a} f(x) dx = \begin{cases} 2\int_0^a f(x) dx, \text{ if } f(x) \text{ is even function or } f(-x) = f(x) \\ 0 \quad, \text{if } f(x) \text{ is odd is odd function or } f(-x) = -f(x) \end{cases}$

This property is generally used when integrand is either even or odd function of x.

Example: 14 The integral $\int_{-1/2}^{1/2} \left([x] + \ln\left(\frac{1+x}{1-x}\right)\right) dx$ equal to

(a) $\frac{-1}{2}$ (b) 0 (c) 1 (d) $2\ln\left(\frac{1}{2}\right)$

Solution: (a) $\log\left(\frac{1+x}{1-x}\right)$ is an odd function of x as $f(-x) = -f(x)$

$I = \int_{-1/2}^{1/2} [x] dx + 0 \Rightarrow I = \int_{-1/2}^{0} [x] dx + \int_0^{1/2} [x] dx \Rightarrow I = \int_{-1/2}^{0} -1 dx + 0 \Rightarrow -[x]_{-1/2}^{0} = \frac{-1}{2}$.

Example: 15 The value of the integral $\int_{-1}^{1} \log\left[x+\sqrt{x^2+1}\right]dx$ is **[MP PET 2001]**

(a) 0 (b) log 2 (c) log 1/2 (d) None of these

Solution: (a) $f(x)=\log[x+\sqrt{x^2+1}]$ is a odd function *i.e.* $f(-x)=-f(x) \Rightarrow f(x)+f(-x)=0 \Rightarrow I=0$.

Example: 16 The value of $\int_{-\pi}^{\pi}(1-x^2)\sin x \cos^2 x dx$ is **[Roorkee 1992; MNR 1998]**

(a) 0 (b) 1 (c) 2 (d) 3

Solution: (a) Let, $f_1(x)=(1-x^2)$, $f_2(x)=\sin x$ and $f_3(x)=\cos^2 x$

Now, $f_1(x)=f_1(-x)$, $f_2(x)=-f_2(-x)$ and $f_3(x)=f_3(-x)$

$\therefore\ I=\int_{-\pi}^{\pi} f(x)dx=\int_{-\pi}^{\pi}[f_1(x).f_2(x).f_3(x)]dx = -\int_{-\pi}^{\pi}[f_1(-x).f_2(-x).f_3(-x)]dx$

$\therefore\ I=0$

(6) $\int_0^{2a} f(x)dx=\int_0^a f(x)dx+\int_0^a f(2a-x)dx$

In particular, $\int_0^{2a} f(x)dx=\begin{cases}0, & \text{if } f(2a-x)=-f(x)\\ 2\int_0^a f(x)dx, & \text{if } f(2a-x)=f(x)\end{cases}$

It is generally used to make half the upper limit.

Example: 17 If n is any integer, then $\int_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)x\,dx$ is equal to **[IIT 1985; Rajasthan PET 1995]**

(a) x (b) 1 (c) 0 (d) None of these

Solution: (c) $I=\int_0^{\pi} e^{\cos^2(\pi-x)}.\cos^3(2n+1)(\pi-x)dx$

$\Rightarrow I=-\int_0^{\pi} e^{\cos^2 x}.\cos^3(2n+1)x\,dx \Rightarrow I=-I$

$\Rightarrow 2I=0 \Rightarrow I=0$.

Example: 18 If $I_1=\int_0^{3\pi} f(\cos^2 x)dx$ and $I_2=\int_0^{\pi} f(\cos^2 x)dx$ then

(a) $I_1=I_2$ (b) $I_1=2I_2$ (c) $I_1=3I_2$ (d) $I_1=4I_2$

Solution: (c) $f(\cos^2 x)=f(\cos^2(3\pi-x))$

$\therefore\ I_1=3\int_0^{\pi} f(\cos^2 x)dx \Rightarrow I_1=3I_2$

(7) $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

Note: ❑ $\int_a^b \frac{f(x)dx}{f(x)+f(a+b-x)}=\frac{1}{2}(b-a)$

Example: 19 $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ is equal to **[IIT 1999]**

(a) 2 (b) –2 (c) 1/2 (d) –1/2

Solution: (a) $I=\int_{\pi/4}^{3\pi/4}\frac{1}{1-\cos x}dx$ $\left[\because \left[\cos\left(\frac{\pi}{4}+\frac{3\pi}{4}-x\right)=-\cos x\right]\right.$

$\therefore\ 2I=\int_{\pi/4}^{3\pi/4}\frac{2}{1-\cos^2 x}dx$

$\Rightarrow 2I=2\int_{\pi/4}^{3\pi/4} cosec^2 x dx \Rightarrow 2I=-2[\cot x]_{\pi/4}^{3\pi/4}=4 \Rightarrow I=2.$

Example: 20 The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$ is **[IIT 1994; Kurukshetra CEE 1998]**

(a) 1 (b) 0 (c) –1 (d) 1/2

Solution: (d) $I=\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$

Put $x=2+3-t \Rightarrow dx=-dt$

$\therefore\quad I=\int_3^2 \frac{\sqrt{5-t}}{\sqrt{5-t}+\sqrt{t}}(-dt)=\int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$ and $2I=\int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}\,dx=\int_2^3 1\,dx$

$\Rightarrow\quad 2I=[x]_2^2=1 \Rightarrow I=1/2$

Example: 21 If $f(a+b-x)=f(x)$ then $\int_a^b x\,f(x)dx$ is equal to **[Kurukshetra CEE 1993; AIEEE 2003]**

(a) $\frac{a+b}{2}\int_a^b f(b-x)dx$ (b) $\frac{a+b}{2}\int_a^b f(x)dx$ (c) $\frac{b-a}{2}\int_a^b f(x)dx$ (d) None of these

Solution: (b) $I=\int_a^b x\,f(x)dx$ and $I=\int_a^b (a+b-x)f(a+b-x)dx$

$\Rightarrow I=\int_a^b (a+b-x)f(x)dx \Rightarrow I=\int_a^b (a+b)f(x)dx-\int_a^b x\,f(x)dx \Rightarrow 2I=\left[\int_a^b f(x)dx\right](a+b) \Rightarrow I=\frac{a+b}{2}\int_a^b f(x)dx$

(8) $\int_0^a x\,f(x)dx=\frac{1}{2}a\int_0^a f(x)dx$ if $f(a-x)=f(x)$

Example: 22 If $\int_0^\pi x\,f(\sin x)dx=k\int_0^\pi f(\sin x)dx$, then the value of k will be **[IIT 1982]**

(a) π (b) $\pi/2$ (c) $\pi/4$ (d) 1

Solution: (b) Given, $\int_0^\pi x\,f(\sin x)dx=k\int_0^\pi f(\sin x)dx$

$\Rightarrow \int_0^\pi (\pi-x)f(\sin(\pi-x))dx=k\int_0^\pi f(\sin(\pi-x))dx \Rightarrow \pi\int_0^\pi f(\sin x)dx-\int_0^\pi x\,f(\sin x)dx=k\int_0^\pi f(\sin x)dx$

$\Rightarrow \pi\int_0^\pi f(\sin x)dx-2k\int_0^\pi f(\sin x)dx=0 \Rightarrow (\pi-2k)\int_0^\pi f(\sin x)dx=0$

$\therefore\ \pi-2k=0 \Rightarrow k=\pi/2$.

(9) If $f(x)$ is a periodic function with period T, then $\int_0^{nT} f(x)dx=n\int_0^T f(x)dx$,

Deduction : If $f(x)$ is a periodic function with period T and $a\in R^+$, then $\int_{nT}^{a+nT} f(x)dx=\int_0^a f(x)dx$

(10) (i) If $f(x)$ is a periodic function with period T, then

$$\int_a^{a+nT} f(x)\,dx=n\int_0^T f(x)\,dx \qquad \text{where } n\in I$$

(a) In particular, if a = 0

$$\int_0^{nT} f(x)\,dx=n\int_0^T f(x)\,dx, \qquad \text{where } n\in I$$

(b) If n = 1, $\int_0^{a+T} f(x)\,dx=\int_0^T f(x)\,dx,$

(i) $\int_{mT}^{nT} f(x)\,dx=(n-m)\int_0^T f(x)\,dx,$ where $n,\ m\in I$

(ii) $\int_{a+nT}^{b+nT} f(x)\,dx=\int_a^b f(x)\,dx,$ where $n\,\varepsilon\, I$

(11) If $f(x)$ is a periodic function with period *T*, then $\int_a^{a+T} f(x)$ is independent of *a*.

(12) $\int_a^b f(x)\,dx = (b-a)\int_0^1 f((b-a)x+a)\,dx$

(13) If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)\,dt$ is an even function

(14) If $f(x)$ is an even function, then $\phi(x) = \int_0^x f(t)\,dt$ is on odd function.

Note : ❑ If $f(t)$ is an even function, then for a non zero '*a*', $\int_0^x f(t)dt$ is not necessarily an odd function. It will be odd function if $\int_0^a f(t)\,dt = 0$

Example: 23 For $n > 0, \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$ is equal to **[IIT 1996]**

(a) π^2 (b) $2\pi^2$ (c) $3\pi^2$ (d) $4\pi^2$

Solution: (a) $I = \int_0^{2\pi} \frac{x \sin^{2n} x dx}{\sin^{2n} x + \cos^{2n} x}$ and $I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}(2\pi - x)dx}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)}$ $\left[\because \int_0^a f(x) = \int_0^a f(a-x)\right]$

$\therefore\ 2I = 2\pi \int_0^{2\pi} \frac{\sin^{2n} \pi}{\sin^{2n} x + \cos^{2n} x} dx \Rightarrow I = \pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$

using $\int_0^{nT} f(x) = n\int_0^T f(x)dx$

$\therefore\ I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \Rightarrow I = 4\pi(\pi/4) = \pi^2$.

Example: 24 If $f(x)$ is a continuous periodic function with period *T*, then the integral $I = \int_a^{a+T} f(x)dx$ is

(a) Equal to $2a$ (b) Equal to $3a$ (c) Independent of *a* (d) None of these

Solution: (c) Consider the function $g(a) = \int_a^{a+T} f(x)dx = \int_a^0 f(x)dx + \int_0^T f(x)dx + \int_T^{a+T} f(x)dx$

Putting $x - T = y$ in last integral, we get $\int_T^{a+T} f(x)dx = \int_0^a f(y+T)dy = \int_0^a f(y)dy$

$\Rightarrow g(a) = \int_a^0 f(x)dx + \int_0^T f(x)dx + \int_0^a f(x)dx = \int_0^T f(x)dx$

Hence $g(a)$ is independent of *a*.

Important Tips

- *Every continuous function defined on [a, b] is integrable over [a, b].*
- *Every monotonic function defined on [a, b] is integrable over [a, b].*
- *If f(x) is a continuous function defined on [a, b], then there exists* $c \in (a,b)$ *such that* $\int_a^b f(x)dx = f(c).(b-a)$.

  The number $f(c) = \frac{1}{(b-a)}\int_a^b f(x)dx$ *is called the mean value of the function* $f(x)$ *on the interval [a, b].*
- *If f is continuous on [a, b], then the integral function g defined by g(x) =* $\int_a^x f(t)dt$ *for* $x \in [a,b]$ *is derivable on [a, b] and* $g'(x) = f(x)$ *for all* $x \in [a,b]$.

☞ *If m and M are the smallest and greatest values of a function f(x) on an interval [a, b], then* $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

☞ *If the function* $\varphi(x)$ *and* $\psi(x)$, *are defined on [a, b] and differentiable at a point* $x \varepsilon (a,b)$ *and f(t) is continuous for* $\phi(a) \le t \le \psi(b)$, *then* $\left(\int_{\varphi(x)}^{\psi(x)} f(t)dt\right) = f(\psi(x))\psi'(x) - f(\varphi(x))\varphi'(x)$

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\, dx$$

☞ *If* $f^2(x)$ *and* $g^2(x)$ *are integrable on [a, b], then* $\left|\int_a^b f(x)g(x)dx\right| \le \left(\int_a^b f^2(x)dx\right)^{1/2}\left(\int_a^b g^2(x)dx\right)^{1/2}$

☞ ***Change of variables*** *: If the function f(x) is continuous on [a, b] and the function* $x = \varphi(t)$ *is continuously differentiable on the interval* $[t_1, t_2]$ *and* $a = \varphi(t_1), b = \varphi(t_2)$, *then* $\int_a^b f(x)dx = \int_{t_1}^{t_2} f(\varphi(t))\varphi'(t)dt$.

☞ *Let a function* $f(x,\alpha)$ *be continuous for* $a \le x \le b$ *and* $c \le \alpha \le d$. *Then for any* $\alpha \in [c,d]$, *if* $I(\alpha) = \int_a^b f(x,\alpha)dx$, *then* $I'(\alpha) = \int_a^b f'(x,\alpha)dx$,

Where $I'(\alpha)$ *is the derivative of* $I(\alpha)$ *w.r.t.* α *and* $f'(x,\alpha)$ *is the derivative of* $f(x,\alpha)$ *w.r.t.* α, *keeping x constant.*

☞ *For a given function* $f(x)$ *continuous on [a, b] if you are able to find two continuous function* $f_1(x)$ *and* $f_2(x)$ *on [a, b] such that* $f_1(x) \le f(x) \le f_2(x)\ \forall x \in [a,b]$, *then* $\int_a^b f_1(x)dx \le \int_a^b f(x)dx \le \int_a^b f_2(x)dx$

6.5 Summation of Series by Integration

We know that $\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=1}^{n} f(a+rh)$, where $nh = b-a$

Now, put $a = 0$, $b = 1$, $\therefore\ nh = 1$ or $h = \frac{1}{n}$. Hence $\int_0^1 f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum f\left(\frac{r}{n}\right)$

Note : ❑ Express the given series in the form $\Sigma \frac{1}{n} f\left(\frac{r}{h}\right)$. Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and the limit of the sum is $\int_0^1 f(x)dx$.

Example: 25 If $S_n = \frac{1}{1+\sqrt{n}} + \frac{1}{2+\sqrt{2n}} + ... + \frac{1}{n+\sqrt{n^2}}$ then $\lim_{n\to\infty} S_n$ is equal to **[Roorkee 2000]**

(a) log 2 (b) 2 log 2 (c) 3 log 2 (d) 4 log 2

Solution: (b) $\sum \lim_{n\to\infty} \frac{1}{r+\sqrt{rn}} = \sum \lim_{n\to\infty} \frac{1}{n\left[\frac{r}{n} + \sqrt{\frac{r}{n}}\right]}$

$\therefore \lim_{n\to\infty} S_n = \int_0^1 \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$

$= 2[\log(1+\sqrt{x})]_0^1 = 2\log 2$

Example: 26 $\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}$ or $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$ is equal to **[WB JEE 1989; Kurukshetra CEE 1998]**

(a) e (b) e^{-1} (c) 1 (d) None of these

Solution: (b) Let $A=\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}$

$\Rightarrow \log A=\lim_{n\to\infty}\log\left(\frac{1.2.3......n}{n^n}\right)^{1/n} \Rightarrow \log A=\lim_{n\to\infty}\log\left(\frac{1}{n}.\frac{2}{n}.\frac{3}{n}....\frac{n}{n}\right)^{1/n} \Rightarrow \log A=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left[\log\left(\frac{r}{n}\right)\right]$

$\Rightarrow \log A=\int_0^1\log x dx=[x\log x-x]_0^1 \Rightarrow \log A=-1 \Rightarrow A=e^{-1}$

6.6 Gamma Function

If m and n are non-negative integers, then $\int_0^{\pi/2}\sin^m x\cos^n x dx=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

where $\Gamma(n)$ is called gamma function which satisfied the following properties

$$\Gamma(n+1)=n\Gamma(n)=n! \quad i.e. \quad \Gamma(1)=1 \text{ and } \Gamma(1/2)=\sqrt{\pi}$$

In place of gamma function, we can also use the following formula :

$$\int_0^{\pi/2}\sin^m x\cos^n x dx=\frac{(m-1)(m-3).....(2 \text{ or } 1)(n-1)(n-3).....(2 \text{ or } 1)}{(m+n)(m+n-2)....(2 \text{ or } 1)}$$

It is important to note that we multiply by ($\pi/2$); when both m and n are even.

Example: 27 The value of $\int_0^{\pi/2}\sin^4 x\cos^6 x dx$ = **[Rajasthan PET 1999]**

(a) $3\pi/312$ (b) $5\pi/512$ (c) $3\pi/512$ (d) $5\pi/312$

Solution: (c) $I=\frac{(4-1).(4-3).(6-1).(6-3).(6-5)}{(4+6)(4+6-2)(4+6-4)(4+6-6)(4+6-8)}.\frac{\pi}{2}=\frac{3.1.5.3.1}{10.8.6.4.2.}.\frac{\pi}{2}=\frac{3\pi}{512}$

6.7 Reduction formulae Definite Integration

(1) $\int_0^\infty e^{-ax}\sin bx dx=\frac{b}{a^2+b^2}$ (2) $\int_0^\infty e^{-ax}\cos bx dx=\frac{a}{a^2+b^2}$ (3) $\int_0^\infty e^{-ax}x^n dx=\frac{n!}{a^n+1}$

Example: 28 If $I_n=\int_0^\infty e^{-x}x^{n-1}dx$, then $\int_0^\infty e^{-\lambda x}x^{n-1}dx$ is equal to

(a) λI_n (b) $\frac{1}{\lambda}I_n$ (c) $\frac{I_n}{\lambda^n}$ (d) $\lambda^n I_n$

Solution: (c) Put, $\lambda x=t$, $\lambda dx=dt$, we get,

$$\int_0^\infty e^{-\lambda x}x^{n-1}dx=\frac{1}{\lambda^n}\int_0^\infty e^{-t}t^{n-1}dt=\frac{1}{\lambda^n}\int_0^\infty e^{-x}x^{n-1}dx=\frac{I_n}{\lambda^n}$$

6.8 Walli's Formula

$$\int_0^{\pi/2}\sin^n x dx=\int_0^{\pi/2}\cos^n x dx=\begin{cases}\frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}......\frac{2}{3}, & \text{when } n \text{ is odd}\\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.......\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}, & \text{when } n \text{ is even}\end{cases}$$

$\int_0^{\pi/2} \sin^m x \cos^n dx = \frac{(m-1)(m-3)..........(n-1)(n-3)........}{(m+n)(m+n-2)}$ [If m, n are both odd $+ve$ integers or one odd $+ve$ integer]

$= \frac{(m-1)(m-3)............(n-1)(n-3)}{(m+n)(m+n-2)} \cdot \frac{\pi}{2}$ [If m, n are both $+ve$ integers]

Example: 29 $\int_0^{\pi/2} \sin^7 x dx$ has value **[BIT Ranchi 1999]**

(a) $\frac{37}{184}$ (b) $\frac{17}{45}$ (c) $\frac{16}{35}$ (d) $\frac{16}{45}$

Solution: (c) Using Walli's formula, $\Rightarrow I = \frac{7-1}{7} \cdot \frac{7-3}{7-2} \cdot \frac{7-5}{7-4} = \frac{6.4.2}{7.5.3} = \frac{16}{35}$

6.9 Leibnitz's Rule

(1) If $f(x)$ is continuous and $u(x)$, $v(x)$ are differentiable functions in the interval $[a, b]$, then,

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f\{v(x)\}\frac{d}{dx}\{v(x)\} - f\{u(x)\}\frac{d}{dx}\{u(x)\}$$

(2) If the function $\phi(x)$ and $\psi(x)$ are defined on $[a,b]$ and differentiable at a point $x \in (a,b)$, and $f(x,t)$ is continuous, then, $\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)dt\right] = \int_{\phi(x)}^{\psi(x)} \frac{d}{dx} f(x,t)\,dt + \left\{\frac{d\,\psi(x)}{dx}\right\} f(x,\psi(x)) - \left\{\frac{d\phi(x)}{dx}\right\} f(x,\phi(x))$

Example: 30 Let $f(x) = \int_1^x \sqrt{2-t^2}\, dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are **[IIT 2002]**

(a) ± 1 (b) $\pm\frac{1}{\sqrt{2}}$ (c) $\pm\frac{1}{2}$ (d) 0 and 1

Solution: (a) $f(x) = \int_1^x \sqrt{2-t^2}dt \Rightarrow f'(x) = \sqrt{2-x^2}.1 - \sqrt{2-1}.0 = \sqrt{2-x^2}$

$\therefore x^2 = f'(x) = \sqrt{2-x^2} \Rightarrow x^4 + x^2 - 2 = 0 \Rightarrow (x^2+2)(x^2-1) = 0$

$\therefore x = \pm 1$ (only real).

Example: 31 Let $f:(0,\infty) \to R$ and $f(x) = \int_0^x f(t)dt$. If $f(x^2) = x^2(1+x)$, then $f(4)$ equals **[IIT 2001]**

(a) 5/4 (b) 7 (c) 4 (d) 2

Solution: (c) By definition of $f(x)$ we have $f(x^2) = \int_0^{x^2} f(t)dt = x^2 + x^3$ (given)

Differentiate both sides, $f(x^2).2x + 0 = 2x + 3x^2$

Put, $x = 2 \Rightarrow 4f(4) = 16 \Rightarrow f(4) = 4$

6.10 Integrals with Infinite Limits (Improper Integral)

A definite integral $\int_a^b f(x)dx$ is called an improper integral, if

The range of integration is finite and the integrand is unbounded and/or the range of integration is infinite and the integrand is bounded.

e.g., The integral $\int_0^1 \frac{1}{x^2}dx$ is an improper integral, because the integrand is unbounded on [0, 1]. Infact, $\frac{1}{x^2} \to \infty$ as $x \to 0$. The integral $\int_0^\infty \frac{1}{1+x^2}dx$ is an improper integral, because the range of integration is not finite.

There are following two kinds of improper definite integrals:

(1) **Improper integral of first kind :** A definite integral $\int_a^b f(x)dx$ is called an improper integral of first kind if the range of integration is not finite (*i.e.*, either $a \to \infty$ or $b \to \infty$ or $a \to \infty$ and $b \to \infty$) and the integrand $f(x)$ is bounded on $[a, b]$.

$$\int_1^\infty \frac{1}{x^2}\,dx, \int_0^\infty \frac{1}{1+x^2}dx, \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx, \int_1^\infty \frac{3x}{(1+2x)^3}\,dx$$ are improper integrals of first kind.

Important Tips

- ☞ *In an improper integral of first kind, the interval of integration is one of the following types* $[a, \infty)$, $(-\infty, b]$, $(-\infty, \infty)$.
- ☞ *The improper integral* $\int_a^\infty f(x)dx$ *is said to be convergent, if* $\lim_{k\to\infty}\int_a^k f(x)dx$ *exists finitely and this limit is called the value of the improper integral. If* $\lim_{k\to\infty}\int_a^k f(x)dx$ *is either* $+\infty$ *or* $-\infty$, *then the integral is said to be divergent.*
- ☞ *The improper integral* $\int_{-\infty}^\infty f(x)dx$ *is said to be convergent, if both the limits on the right-hand side exist finitely and are independent of each other. The improper integral* $\int_{-\infty}^\infty f(x)dx$ *is said to be divergent if the right hand side is* $+\infty$ *or* $-\infty$

(2) **Improper integral of second kind :** A definite integral $\int_a^b f(x)dx$ is called an improper integral of second kind if the range of integration $[a, b]$ is finite and the integrand is unbounded at one or more points of $[a, b]$.

If $\int_a^b f(x)dx$ is an improper integral of second kind, then a, b are finite real numbers and there exists at least one point $c \in [a,b]$ such that $f(x) \to +\infty$ or $f(x) \to -\infty$ as $x \to c$ *i.e.*, $f(x)$ has at least one point of finite discontinuity in $[a, b]$.

For example :

(i) The integral $\int_1^3 \frac{1}{x-2}dx$, is an improper integral of second kind, because $\lim_{x\to 2}\left(\frac{1}{x-2}\right) = \infty$.

(ii) The integral $\int_0^1 \log x dx$; is an improper integral of second kind, because $\log x \to \infty$ as $x \to 0$.

(iii) The integral $\int_0^{2\pi} \frac{1}{1+\cos x}dx$, is an improper integral of second kind since integrand $\frac{1}{1+\cos x}$ becomes infinite at $x = \pi \in [0, 2\pi]$.

(iv) $\int_0^1 \frac{\sin x}{x}dx$, is a proper integral since $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Important Tips

- ☞ *Let f(x) be bounded function defined on (a, b] such that a is the only point of infinite discontinuity of f(x) i.e., f(x)* $\to \infty$ *as x* $\to$ *a. Then the improper integral of f(x) on (a, b] is denoted by* $\int_a^b f(x)dx$ *and is defined as* $\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^b f(x)dx$. *Provided that the limit on right hand side exists. If l denotes the limit on right hand side, then the improper integral* $\int_a^b f(x)dx$ *is said to converge to l, when l is finite. If l =* $+\infty$ *or l =* $-\infty$, *then the integral is said to be a divergent integral.*
- ☞ *Let f(x) be bounded function defined on [a, b) such that b is the only point of infinite discontinuity of f(x) i.e. f(x)* $\to \infty$ *as x* $\to$ *b. Then the improper integral of f(x) on [a, b) is denoted by* $\int_a^b f(x)dx$ *and is defined as* $\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx$

Provided that the limit on right hand side exists finitely. If l denotes the limit on right hand side, then the improper integral $\int_a^b f(x)dx$ *is said to converge to l, when l is finite.*

If $l = +\infty$ *or* $l = -\infty$, *then the integral is said to be a divergent integral.*

☞ *Let f(x) be a bounded function defined on (a, b) such that a and b are only two points of infinite discontinuity of f(x) i.e., f(a)* $\to \infty$, *f(b)* $\to \infty$.

Then the improper integral of f(x) on (a, b) is denoted by $\int_a^b f(x)dx$ *and is defined as*

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{c} f(x)dx + \lim_{\delta \to 0} \int_a^{b-\delta} f(x)dx,\ a < c < b$$

Provided that both the limits on right hand side exist.

☞ *Let f(x) be a bounded function defined [a, b]-{c},* $c \in$*[a, b] and c is the only point of infinite discontinuity of f(x) i.e. f(c)*$\to\infty$. *Then the improper integral of f(x) on [a, b] – {c} is denoted by* $\int_a^b f(x)dx$ *and is defined as* $\int_a^b f(x)dx = \lim_{x \to 0} \int_a^{c-x} f(x)dx + \lim_{\delta \to 0} \int_{c+\delta}^{b} f(x)dx$

Provided that both the limits on right hand side exist finitely. The improper integral $\int_a^b f(x)dx$ *is said to be convergent if both the limits on the right hand side exist finitely.*

☞ *If either of the two or both the limits on RHS are* $\pm\infty$, *then the integral is said to be divergent.*

Example: 32 The improper integral $\int_0^{\infty} e^{-x}dx$ is …… and the value is….

(a) Convergent, 1 (b) Divergent, 1 (c) Convergent, 0 (d) Divergent, 0

Solution: (a) $I = \int_0^{\infty} e^{-x}dx = \lim_{k\to\infty} \int_0^k e^{-x}dx \Rightarrow I = \lim_{k\to\infty}[-e^{-x}]_0^k = \lim_{k\to\infty}[-e^{-k} + e^0] \Rightarrow I = \lim_{k\to\infty}(1 - e^{-k}) = 1 - 0 = 1$ $[\because \lim_{k\to\infty} e^{-k} = e^{-\infty} = 0]$

Thus, $\lim_{k\to\infty} \int_0^k e^{-x}dx$ exists and is finite. Hence the given integral is convergent.

Example: 33 The integral $\int_{-\infty}^0 \frac{1}{a^2 + x^2} dx, a \neq 0$ is

(a) Convergent and equal to $\frac{\pi}{a}$ (b) Convergent and equal to $\frac{\pi}{2a}$

(c) Divergent and equal to $\frac{\pi}{a}$ (d) Divergent and equal to $\frac{\pi}{2a}$

Solution: (b) $I = \int_{-\infty}^0 \frac{dx}{a^2 + x^2} = \lim_{k\to-\infty} \int_k^0 \frac{dx}{a^2 + x^2}$

$\Rightarrow I = \lim_{k\to-\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_k^0 = \lim_{k\to-\infty}\left[\frac{1}{a}\tan^{-1} 0 - \frac{1}{a}\tan^{-1}\frac{k}{a}\right] \Rightarrow I = 0 - \frac{1}{a}\tan^{-1}(-\infty) = -\frac{1}{a}\left(\frac{-\pi}{2}\right) = \frac{\pi}{2a}$

Hence integral is convergent.

Example: 34 The integral $\int_{-\infty}^{\infty} \frac{1}{e^{-x} + e^x} dx$ is

(a) Convergent and equal to $\pi/6$ (b) Convergent and equal to $\pi/4$

(c) Convergent and equal to $\pi/3$ (d) Convergent and equal to $\pi/2$

Solution: (d) $I = \int_{-\infty}^{\infty} \frac{1}{e^{-x} + e^x} dx = \int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}} dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$\therefore\ I = \int_0^{\infty} \frac{1}{1 + t^2} dt \Rightarrow I = [\tan^{-1} t]_0^{\infty} = [\tan^{-1}\infty - \tan^{-1} 0] \Rightarrow I = \pi/2$, which is finite so convergent.

Example: 35 $\int_1^2 \frac{x+1}{\sqrt{x-1}} dx$ is

(a) Convergent and equal to $\frac{14}{3}$ (b) Divergent and equal to $\frac{3}{14}$

(c) Convergent and equal to ∞ (d) Divergent and equal to ∞

Solution: (a) $I=\int_1^2 \sqrt{x-1}dx+\int_1^2 \frac{2}{\sqrt{x-1}}dx = \left[\frac{2}{3}(x-1)^{3/2}\right]_1^2 + [4\sqrt{x-1}]_1^2 = 14/3$ which is finite so convergent.

Example: 36 $\int_1^2 \frac{dx}{x^2-5x+4}dx$ is

(a) Convergent and equal to $\frac{1}{3}\log 2$ (b) Convergent and equal to 3/log2

(c) Divergent (d) None of these

Solution: (c) $I=\int_1^2 \frac{dx}{(x-1)(x-4)} = \frac{1}{3}\int_1^2\left(\frac{1}{x-4}-\frac{1}{x-1}\right)dx = \frac{1}{3}[\log 2-\infty]=-\infty$

So the given integral is not convergent.

6.11 Some Important results of Definite Integral

(1) If $I_n=\int_0^{\pi/4}\tan^n x dx$ then $I_n+I_{n-2}=\frac{1}{n-1}$

(2) If $I_n=\int_0^{\pi/4}\cot^n x dx$ then $I_n+I_{n-2}=\frac{1}{1-n}$

(3) If $I_n=\int_0^{\pi/4}\sec^n x dx$ then $I_n=\frac{(\sqrt{2})^{n-2}}{n-1}+\frac{n-2}{n-1}I_{n-2}$

(4) If $I_n=\int_0^{\pi/4}\text{cosec}^{\text{n}} x dx$ then $I_n=\frac{(\sqrt{2})^{n-2}}{n-1}+\frac{n-2}{n-1}I_{n-2}$

(5) If $I_n=\int_0^{\pi/2}x^n \sin x dx$ then $I_n+n(n-1)I_{n-2}=n(\pi/2)^{n-1}$

(6) If $I_n=\int_0^{\pi/2}x^n \cos x dx$ then $I_n+n(n-1)I_{n-2}=(\pi/2)^n$

(7) If $a>b>0,$ then $\int_0^{\pi/2}\frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a+b}{a-b}}$

(8) If $0<a<b$ then $\int_0^{\pi/2}\frac{dx}{a+b\cos x}=\frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}-\sqrt{b-a}}{\sqrt{b+a}+\sqrt{b-a}}\right|$

(9) If $a>b>0$ then $\int_0^{\pi/2}\frac{dx}{a+b\sin x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}$

(10) If $0<a<b$, then $\int_0^{\pi/2}\frac{dx}{a+b\sin x}=\frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}+\sqrt{b-a}}{\sqrt{b+a}-\sqrt{b-a}}\right|$

(11) If $a>b, a^2>b^2+c^2,$ then $\int_0^{\pi/2}\frac{dx}{a+b\cos x+c\sin x}=\frac{2}{\sqrt{a^2-b^2-c^2}}\tan^{-1}\frac{a-b+c}{\sqrt{a^2-b^2-c^2}}$

(12) If $a>b, a^2<b^2+c^2,$ then $\int_0^{\pi/2}\frac{dx}{a+b\cos x+c\sin x}=\frac{1}{\sqrt{b^2+c^2-a^2}}\log\left|\frac{a-b+c-\sqrt{b^2+c^2-a^2}}{a-b+c+\sqrt{b^2+c^2-a^2}}\right|$

(13) If $a < b,\ a^2 < b^2 + c^2$ then $\int_0^{\pi/2} \frac{dx}{a + b\cos x + c\sin x} = \frac{-1}{\sqrt{b^2 + c^2 - a^2}} \log\left|\frac{b - a - c - \sqrt{b^2 + c^2 - a^2}}{b - a - c + \sqrt{b^2 + c^2 - a^2}}\right|$

Important Tips

☞ $\lim_{x \to 0} \left|\frac{\int_0^x f(x)dx}{x}\right| = f(0)$

☞ $\int_a^b f(x)dx = (b-a)\int_0^1 f[(b-a)t + a]dt$

6.12 Integration of Piecewise Continuous Functions

Any function $f(x)$ which is discontinuous at finite number of points in an interval $[a, b]$ can be made continuous in sub-intervals by breaking the intervals into these subintervals. If $f(x)$ is discontinuous at points $x_1,\ x_2,\ x_3..........x_n$ in (a, b), then we can define subintervals $(a, x_1), (x_1, x_2)............(x_{n-1},\ x_n), (x_n, b)$ such that $f(x)$ is continuous in each of these subintervals. Such functions are called piecewise continuous functions. For integration of Piecewise continuous function. We integrate $f(x)$ in these sub-intervals and finally add all the values.

Example: 37 $\int_{-10}^{20} [\cot^{-1} x]\,dx$, where [.] denotes greatest integer function

(a) $30 + \cot 1 + \cot 3$ (b) $30 + \cot 1 + \cot 2 + \cot 3$

(c) $8\ 30 + \cot 1 + \cot 2$ (d) None of these

Solution: (b) Let $I = \int_{-10}^{20} [\cot^{-1} x]\,dx$,

we know $\cot^{-1} x \in (0, \pi)\ \forall\ x \in R$

thus, $[\cot^{-1} x] = \begin{cases} 3, & x \in (-\infty, \cot 3) \\ 2, & x \in (\cot 3, \cot 2) \\ 1 & x \in (\cot 2, \cot 1) \\ 0 & x \in (\cot 1, \infty) \end{cases}$

Hence, $I = \int_{-10}^{\cot 3} 3\,dx + \int_{\cot 3}^{\cot 2} 2\,dx + \int_{\cot 2}^{\cot 1} 1\,dx + \int_{\cot 1}^{20} 0\,dx = 30 + \cot 1 + \cot 2 + \cot 3$

Example: 38 $\int_0^2 [x^2 - x + 1]\,dx$, where [.] denotes greatest integer function

(a) $\frac{7 - \sqrt{5}}{2}$ (b) $\frac{7 + \sqrt{5}}{2}$ (c) $\frac{\sqrt{5} - 3}{2}$ (d) None of these

Solution: (a) Let $I = \int_0^2 [x^2 - x + 1]\,dx = \int_0^{\frac{1+\sqrt{5}}{2}} [x^2 - x + 1]\,dx + \int_{\frac{1+\sqrt{5}}{2}}^2 [x^2 - x + 1]\,dx = \int_0^{\frac{1+\sqrt{5}}{2}} 1\,dx + \int_{\frac{1+\sqrt{5}}{2}}^2 2dx = \frac{7 - \sqrt{5}}{2}$