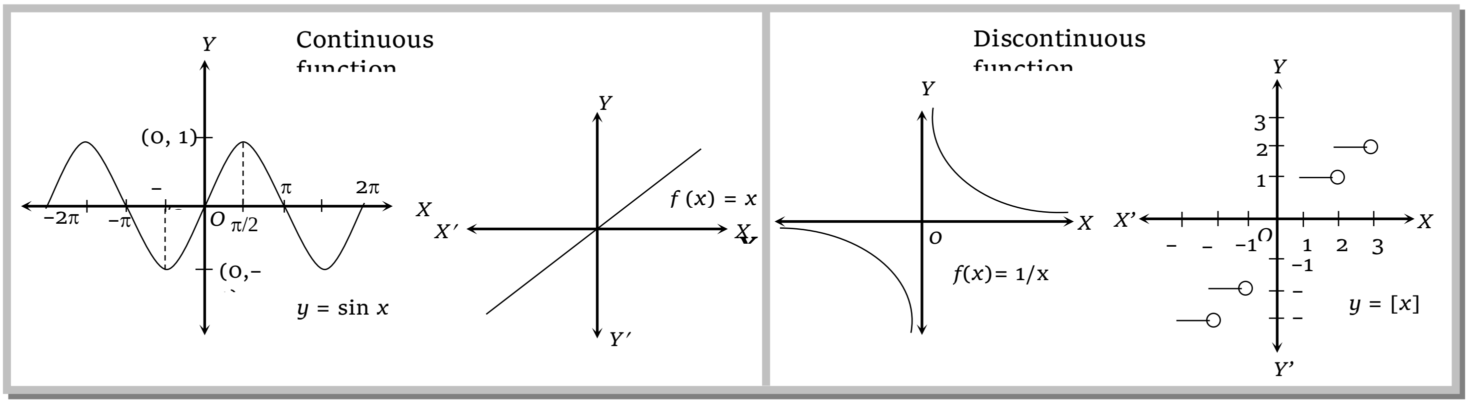
2.3 Continuity

Introduction

The word 'Continuous' means without any break or gap. If the graph of a function has no break, or gap or jump, then it is said to be continuous.

A function which is not continuous is called a discontinuous function.

While studying graphs of functions, we see that graphs of functions $\sin x$, x, $\cos x$, e^x etc. are continuous but greatest integer function $[x]$ has break at every integral point, so it is not continuous. Similarly $\tan x$, $\cot x$, $\sec x$, $\frac{1}{x}$ etc. are also discontinuous function.



2.3.1 Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain iff $\lim_{x \to a} f(x) = f(a)$. *i.e.* a function $f(x)$ is continuous at $x = a$ if and only if it satisfies the following three conditions :

(1) $f(a)$ exists. ('a' lies in the domain of f)

(2) $\lim_{x \to a} f(x)$ exist *i.e.* $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ or R.H.L. = L.H.L.

(3) $\lim_{x \to a} f(x) = f(a)$ (limit equals the value of function).

Cauchy's definition of continuity : A function f is said to be continuous at a point a of its domain D if for every $\varepsilon > 0$ there exists $\delta > 0$ (dependent on ε) such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Comparing this definition with the definition of limit we find that $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists and is equal to $f(a)$ *i.e.*, if $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a+} f(x)$.

Heine's definition of continuity : A function f is said to be continuous at a point a of its domain D, converging to a, the sequence $< a_n >$ of the points in D converging to a, the sequence $< f(a_n) >$ converges to $f(a)$ *i.e.* $\lim a_n = a \Rightarrow \lim f(a_n) = f(a)$. This definition is mainly used to prove the discontinuity to a function.

Note : ❒ Continuity of a function at a point, we find its limit and value at that point, if these two exist and are equal, then function is continuous at that point.

Formal definition of continuity : The function $f(x)$ is said to be continuous at $x = a$, in its domain if for any arbitrary chosen positive number $\in > 0$, we can find a corresponding number δ depending on $\in$ such that $|f(x) - f(a)| < \in \ \forall x$ for which $0 < |x - a| < \delta$.

2.3.2 Continuity from Left and Right

Function $f(x)$ is said to be

(1) Left continuous at $x = a$ if $\lim_{x \to a-0} f(x) = f(a)$

(2) Right continuous at $x = a$ if $\lim_{x \to a+0} f(x) = f(a)$.

Thus a function $f(x)$ is continuous at a point $x = a$ if it is left continuous as well as right continuous at $x = a$.

Example: 1 If $f(x) = \begin{cases} x + \lambda, & x < 3 \\ 4, & x = 3 \\ 3x - 5, & x > 3 \end{cases}$ is continuous at $x = 3$, then λ =

(a) 4 (b) 3 (c) 2 (d) 1

Solution: (d) L.H.L. at $x = 3$, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x + \lambda) = \lim_{h \to 0} (3 - h + \lambda) = 3 + \lambda$ .....(i)

R.H.L. at $x = 3$, $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (3x - 5) = \lim_{h \to 0} \{3(3 + h) - 5\} = 4$ .....(ii)

Value of function $f(3) = 4$ .....(iii)

For continuity at $x = 3$

Limit of function = value of function $3 + \lambda = 4 \Rightarrow \lambda = 1$.

Example: 2 If $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is **[MP PET 1999; AMU 1999; Rajasthan PET 20**

(a) 1 (b) –1 (c) 0 (d) 2

Solution: (c) If function is continuous at $x = 0$, then by the definition of continuity $f(0) = \lim_{x \to 0} f(x)$

since $f(0) = k$. Hence, $f(0) = k = \lim_{x \to 0} (x) \left(\sin \frac{1}{x} \right)$

$\Rightarrow k = 0$ (a finite quantity lies between –1 to 1) $\Rightarrow k = 0$.

Example: 3 If $f(x) = \begin{cases} 2x + 1 & \text{when } x < 1 \\ k & \text{when } x = 1 \\ 5x - 2 & \text{when } x > 1 \end{cases}$ is continuous at $x = 1$, then the value of k is **[Rajasthan PET 2001]**

(a) 1 (b) 2 (c) 3 (d) 4

Solution: (c) Since $f(x)$ is continuous at $x = 1$,

$\Rightarrow \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$ .....(i)

Now $\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} 2(1 - h) + 1 = 3$ *i.e.,* $\lim_{x \to 1^-} f(x) = 3$

Similarly, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 5(1+h)-2$ *i.e.*, $\lim_{x\to 1^+} f(x) = 3$

So according to equation (i), we have k = 3.

Example: 4 The value of k which makes $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ k, & x = 0 \end{cases}$ continuous at x = 0 is **[Rajasthan PET 1993; UPSEAT 1995]**

(a) 8 (b) 1 (c) –1 (d) None of these

Solution: (d) We have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin\frac{1}{x}$ = An oscillating number which oscillates between –1 and 1.

Hence, $\lim_{x\to 0} f(x)$ does not exist. Consequently $f(x)$ cannot be continuous at $x = 0$ for any value of k.

Example: 5 The value of m for which the function $f(x) = \begin{cases} mx^2, x \leq 1 \\ 2x, x > 1 \end{cases}$ is continuous at $x = 1$, is

(a) 0 (b) 1 (c) 2 (d) Does not exist

Solution: (c) LHL = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} m(1-h)^2 = m$

RHL = $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} 2(1+h) = 2$ and $f(1) = m$

Function is continuous at $x = 1$, $\therefore$ LHL = RHL = $f(1)$

Therefore $m = 2$.

Example: 6 If the function $f(x) = \begin{cases} (\cos x)^{1/x}, x \neq 0 \\ k \quad\quad , x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is

(a) 1 (b) –1 (c) 0 (d) e

Solution: (a) $\lim_{x\to 0} (\cos x)^{1/x} = k \Rightarrow \lim_{x\to 0} \frac{1}{x} \log(\cos x) = \log k \Rightarrow \lim_{x\to 0} \frac{1}{x} \lim_{x\to 0} \log \cos x = \log k \Rightarrow \lim_{x\to 0} \frac{1}{x} \times 0 = \log_e k \Rightarrow k = 1$.

2.3.3 Continuity of a Function in Open and Closed Interval

Open interval : A function $f(x)$ is said to be continuous in an open interval (a, b) iff it is continuous at every point in that interval.

Note : ❑ This definition implies the non-breakable behavior of the function $f(x)$ in the interval (a, b).

Closed interval : A function $f(x)$ is said to be continuous in a closed interval [a, b] iff,

(1) f is continuous in (a, b)

(2) f is continuous from the right at 'a' *i.e.* $\lim_{x\to a^+} f(x) = f(a)$

(3) f is continuous from the left at 'b' *i.e.* $\lim_{x\to b^-} f(x) = f(b)$.

Example: 7 If the function $f(x)=\begin{cases} x+a^2\sqrt{2}\sin x & , \quad 0\le x<\frac{\pi}{4} \\ x\cot x+b & , \quad \frac{\pi}{4}\le x<\frac{\pi}{2} \\ b\sin 2x-a\cos 2x & , \quad \frac{\pi}{2}\le x\le \pi \end{cases}$, is continuous in the interval [0, π] then the values of (a, b) are

[Roorkee 1998]

(a) (–1, –1) (b) (0, 0) (c) (–1, 1) (d) (1, –1)

Solution: (b) Since f is continuous at $x=\frac{\pi}{4}$; $\therefore f\left(\frac{\pi}{4}\right)=\underset{h\to 0}{f}\left(\frac{\pi}{4}+h\right)=\underset{h\to 0}{f}\left(\frac{\pi}{4}-h\right)\Rightarrow \frac{\pi}{4}(1)+b=\left(\frac{\pi}{4}-0\right)+a^2\sqrt{2}\sin\left(\frac{\pi}{4}-0\right)$

$\Rightarrow \frac{\pi}{4}+b=\frac{\pi}{4}+a^2\sqrt{2}\sin\frac{\pi}{4}\Rightarrow b=a^2\sqrt{2}.\frac{1}{\sqrt{2}}\Rightarrow b=a^2$

Also as f is continuous at $x=\frac{\pi}{2}$; $\therefore f\left(\frac{\pi}{2}\right)=\lim_{x\to\frac{\pi}{2}-0} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$

$\Rightarrow b\sin 2\frac{\pi}{2}-a\cos 2\frac{\pi}{2}=\lim_{h\to 0}\left[\left(\frac{\pi}{2}-h\right)\cot\left(\frac{\pi}{2}-h\right)+b\right]\Rightarrow b.0-a(-1)=0+b\Rightarrow a=b$.

Hence (0, 0) satisfy the above relations.

Example: 8 If the function $f(x)=\begin{cases} 1+\sin\frac{\pi x}{2} & \text{for } -\infty<x\le 1 \\ ax+b & \text{for } 1<x<3 \\ 6\tan\frac{x\pi}{12} & \text{for } 3\le x<6 \end{cases}$ is continuous in the interval $(-\infty, 6)$ then the values of a and b are respectively **[MP PET 1998]**

(a) 0, 2 (b) 1, 1 (c) 2, 0 (d) 2, 1

Solution: (c) $\because$ The turning points for $f(x)$ are $x=1, 3$.

So, $\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h)=\lim_{h\to 0}\left[1+\sin\frac{\pi}{2}(1-h)\right]=\left[1+\sin\left(\frac{\pi}{2}-0\right)\right]=2$

Similarly, $\lim_{x\to 1^+} f(x)=\lim_{h\to 0} f(1+h)=\lim_{h\to 0} a(1+h)+b=a+b$

$\because f(x)$ is continuous at $x=1$, so $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$

$\Rightarrow 2=a+b$ ....(i)

Again, $\lim_{x\to 3^-} f(x)=\lim_{h\to 0} f(3-h)=\lim_{h\to 0} a(3-h)+b=3a+b$ and $\lim_{x\to 3^+} f(x)=\lim_{h\to 0} f(3+h)=\lim_{h\to 0} 6\tan\frac{\pi}{12}(3+h)=6$

$f(x)$ is continuous in $(-\infty, 6)$, so it is continuous at $x=3$ also, so $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3)$

$\Rightarrow 3a+b=6$ ......(ii)

Solving (i) and (ii) $a=2$, $b=0$.

Trick : In above type of questions first find out the turning points. For example in above question they are x = 1,3. Now find out the values of the function at these points and if they are same then the function is continuous *i.e.*, in above problem.

$$f(x)=\begin{cases} 1+\sin\frac{\pi}{2}x & ; \quad -\infty<x\le 1, & f(1)=2 \\ ax+b & ; \quad 1<x<3 & f(1)=a+b,\ f(3)=3a+b \\ 6\tan\frac{\pi x}{12} & ; \quad 3\le x<6 & f(3)=6 \end{cases}$$

Which gives $2=a+b$ and $6=3a+b$ after solving above linear equations we get $a=2,\ b=0$.

Example: 9 If $f(x)=\begin{cases} x\sin x, & \text{when } 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}\sin(\pi+x), & \text{when } \frac{\pi}{2} < x < \pi \end{cases}$ then **[IIT 1991]**

(a) $f(x)$ is discontinuous at $x=\frac{\pi}{2}$ (b) $f(x)$ is continuous at $x=\frac{\pi}{2}$

(c) $f(x)$ is continuous at $x=0$ (d) None of these

Solution: (a) $\lim_{x\to\frac{\pi}{2}^-} f(x)=\frac{\pi}{2}$, $\lim_{x\to\frac{\pi}{2}^+} f(x)=-\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$.`

Since $\lim_{x\to\frac{\pi}{2}^-} \ne \lim_{x\to\frac{\pi}{2}^+} f(x)$, $\therefore$ Function is discontinuous at $x=\frac{\pi}{2}$

Example: 10 If $f(x)=\begin{cases} \frac{1-\cos 4x}{x^2}, & \text{when } x<0 \\ a, & \text{when } x=0 \\ \frac{\sqrt{x}}{\sqrt{(16+\sqrt{x})}-4}, & \text{when } x>0 \end{cases}$ is continuous at $x = 0$, then the value of 'a' will be **[IIT 1990; AMU 2**

(a) 8 (b) –8 (c) 4 (d) None of these

Solution: (a) $\lim_{x\to 0^-} f(x) \lim_{x\to 0^-}\left(\frac{2\sin^2 2x}{(2x)^2}\right)4=8$ and $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}[(\sqrt{16+\sqrt{x}})+4]=8$

Hence $a=8$.

2.3.4 Continuous Function

(1) **A list of continuous functions :**

Function $f(x)$	Interval in which $f(x)$ is continuous
(i) Constant K	$(-\infty, \infty)$
(ii) x^n, (n is a positive integer)	$(-\infty, \infty)$
(iii) x^{-n} (n is a positive integer)	$(-\infty, \infty)-\{0\}$
(iv) $\|x-a\|$	$(-\infty, \infty)$
(v) $p(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+........+a_n$	$(-\infty, \infty)$
(vi) $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial in x	$(-\infty, \infty)-\{x : q(x)=0\}$
(vii) $\sin x$	$(-\infty, \infty)$
(viii) $\cos x$	$(-\infty, \infty)$
(ix) $\tan x$	$(-\infty, \infty)-\{(2n+1)\pi/2 : n \in I\}$
(x) $\cot x$	$(-\infty, \infty)-\{n\pi : n \in I\}$
(xi) $\sec x$	$(-\infty, \infty)-\{(2n+1)\pi/2 : n \in I\}$

(xii) $\text{cosec}\, x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
(xiii) e^x	$(-\infty, \infty)$
(xiv) $\log_e x$	$(0, \infty)$

(2) **Properties of continuous functions :** Let $f(x)$ and $g(x)$ be two continuous functions at $x = a$. Then

(i) $cf(x)$ is continuous at $x = a$, where c is any constant

(ii) $f(x) \pm g(x)$ is continuous at $x = a$.

(iii) $f(x) \,.\, g(x)$ is continuous at $x = a$.

(iv) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.

Important Tips

☞ *A function $f(x)$ is said to be continuous if it is continuous at each point of its domain.*

☞ *A function $f(x)$ is said to be everywhere continuous if it is continuous on the entire real line R i.e. $(-\infty, \infty)$. eg. polynomial function e^x, $\sin x, \cos x$, constant, x^n, $|x - a|$ etc.*

☞ *Integral function of a continuous function is a continuous function.*

☞ *If g(x) is continuous at x = a and f(x) is continuous at x = g(a) then (fog) (x) is continuous at $x = a$.*

☞ *If f(x) is continuous in a closed interval [a, b] then it is bounded on this interval.*

☞ *If f(x) is a continuous function defined on [a, b] such that f(a) and f(b) are of opposite signs, then there is atleast one value of x for which f(x) vanishes. i.e. if f(a)>0, f(b) < 0 $\Rightarrow \exists c \in (a, b)$ such that f(c) = 0.*

☞ *If f(x) is continuous on [a, b] and maps [a, b] into [a, b] then for some x $\in$ [a, b] we have f(x) = x.*

(3) **Continuity of composite function :** If the function $u = f(x)$ is continuous at the point $x = a$, and the function $y = g(u)$ is continuous at the point $u = f(a)$, then the composite function $y = (gof)(x) = g(f(x))$ is continuous at the point $x = a$.

2.3.5 Discontinuous Function

(1) **Discontinuous function :** A function 'f' which is not continuous at a point $x = a$ in its domain is said to be discontinuous there at. The point 'a' is called a point of discontinuity of the function.

The discontinuity may arise due to any of the following situations.

(i) $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ or both may not exist

(ii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ may exist, but are unequal.

(iii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ both may exist, but either of the two or both may not be equal to $f(a)$.

Important Tips

☞ *A function f is said to have removable discontinuity at x = a if $\lim_{x+a^+} f(x) = \lim_{x+a^-} f(x)$ but their common value is not equal to f(a).*

Such a discontinuity can be removed by assigning a suitable value to the function f at x = a.

- *If $\lim_{x\to a} f(x)$ does not exist, then we can not remove this discontinuity. So this become a non-removable discontinuity or essential discontinuity.*
- *If f is continuous at x = c and g is discontinuous at x = c, then*
  (a) f +g and f – g are discontinuous (b) f.g may be continuous
- *If f and g are discontinuous at x = c, then f + g, f – g and fg may still be continuous.*
- *Point functions (domain and range consists one value only) is not a continuous function.*

Example: 11 The points of discontinuity of $y = \frac{1}{u^2+u-2}$ where $u = \frac{1}{x-1}$ is

(a) $\frac{1}{2}, 1, 2$ (b) $\frac{-1}{2}, 1, -2$ (c) $\frac{1}{2}, -1, 2$ (d) None of these

Solution: (a) The function $u = f(x) = \frac{1}{x-1}$ is discontinuous at the point $x = 1$. The function

$y = g(x) = \frac{1}{u^2+u-2} = \frac{1}{(u+2)(u-1)}$ is discontinuous at $u = -2$ and $u = 1$

when $u = -2 \Rightarrow \frac{1}{x-1} = -2 \Rightarrow x = \frac{1}{2}$, when $u = 1 \Rightarrow \frac{1}{x-1} = 1 \Rightarrow x = 2$.

Hence, the composite $y = g(f(x))$ is discontinuous at three points $= \frac{1}{2}, 1, 2$.

Example: 12 The function $f(x) = \frac{\log(1+ax) - \log(1-bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$ so that it is continuous at $x = 0$, is

(a) $a-b$ (b) $a+b$ (c) $\log a + \log b$ (d) $\log a - \log b$

Solution: (b) Since limit of a function is $a+b$ as $x \to 0$, therefore to be continuous at $x = 0$, its value must be $a+b$ at $x = 0 \Rightarrow f(0) = a+b$.

Example: 13 If $f(x) = \begin{cases} \frac{x^2-4x+3}{x^2-1}, & \text{for } x \neq 1 \\ 2, & \text{for } x = 1 \end{cases}$, then **[IIT 1972]**

(a) $\lim_{x\to 1^+} f(x) = 2$ (b) $\lim_{x\to 1^-} f(x) = 3$

(c) $f(x)$ is discontinuous at $x = 1$ (d) None of these

Solution: (c) $f(1) = 2,\ f(1+) = \lim_{x\to 1+} \frac{x^2-4x+3}{x^2-1} = \lim_{x\to 1+} \frac{(x-3)}{(x+1)} = -1$

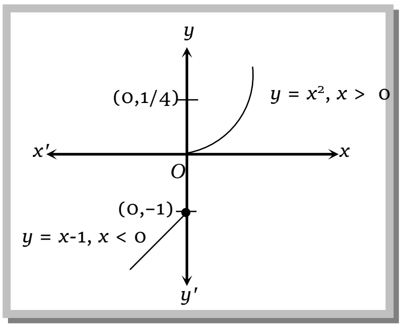
$f(1-) = \lim_{x\to 1-} \frac{x^2-4x+3}{x^2-1} = -1 \Rightarrow f(1) \neq f(1-)$. Hence the function is discontinuous at $x = 1$.

Example: 14 If $f(x) = \begin{cases} x-1, & x < 0 \\ \frac{1}{4}, & x = 0 \\ x^2, & x > 0 \end{cases}$, then **[Roorkee 1988]**

(a) $\lim_{x\to 0^+} f(x) = 1$ (b) $\lim_{x\to 0^-} f(x) = 1$

(c) $f(x)$ is discontinuous at $x = 0$ (d) None of these

Solution: (c) Clearly from curve drawn of the given function $f(x)$, it is discontinuous at $x = 0$.



Example: 15 Let $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} , & -\frac{\pi}{6}<x<0 \\ b , & x=0 \\ e^{\frac{\tan 2x}{\tan 3x}} , & 0<x<\frac{\pi}{6}\end{cases}$, then the values of a and b if f is continuous at $x=0,$ are respectively

(a) $\frac{2}{3},\frac{3}{2}$ (b) $\frac{2}{3},e^{2/3}$ (c) $\frac{3}{2},e^{3/2}$ (d) None of these

Solution: (b) $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} ; & -\left(\frac{\pi}{6}\right)<x<0 \\ b ; & x=0 \\ e^{\frac{\tan 2x}{\tan 3x}} ; & 0<x<\left(\frac{\pi}{6}\right)\end{cases}$

For $f(x)$ to be continuous at $x=0$

$\Rightarrow \lim_{x\to 0^-} f(x)=f(0)=\lim_{x\to 0^+} f(x) \Rightarrow \lim_{x\to 0}(1+|\sin x|)^{\frac{a}{|\sin x|}}=e^{\lim_{x\to 0^-}\left(|\sin x|\frac{a}{|\sin x|}\right)}=e^a$

Now, $\lim_{x\to 0^+} e^{\tan 2x/\tan 3x}=\lim_{x\to 0^+} e^{\left(\frac{\tan 2x}{2x}.2x\right)/\left(\frac{\tan 3x}{3x}.3x\right)}=\lim_{x\to 0^+} e^{2/3}=e^{2/3}.$

$\therefore\ e^a=b=e^{2/3} \Rightarrow a=\frac{2}{3}$ and $b=e^{2/3}$.

Example: 16 Let $f(x)$ be defined for all $x>0$ and be continuous. Let $f(x)$ satisfy $f\left(\frac{x}{y}\right)=f(x)-f(y)$ for all x, y and $f(e)=1,$ then

[IIT 1995]

(a) $f(x)=$ In x (b) $f(x)$ is bounded (c) $f\left(\frac{1}{x}\right)\to 0$ as $x\to 0$ (d) $xf(x)\to 1$ as $x\to 0$

Solution: (a) Let $f(x)=$ In $(x), x>0$ $f(x)=$ In (x) is a continuous function of x for every positive value of x.

$f\left(\frac{x}{y}\right)=$ In $\left(\frac{x}{y}\right)=$ In $(x)-$ In $(y)=f(x)-f(y).$

Example: 17 Let $f(x)=[x]\sin\left(\frac{\pi}{[x+1]}\right)$, where $[.]$ denotes the greatest integer function. The domain of f is ..... and the points of discontinuity of f in the domain are

(a) $\{x \in R |\ x \in [-1, 0)\}, I-\{0\}$ (b) $\{x \in R |\ x \notin [1, 0)\}, I-\{0\}$

(c) $\{x \in R |\ x \notin [-1, 0)\},\ I-\{0\}$ (d) None of these

Solution: (c) Note that $[x+1]=0$ if $0 \le x+1 < 1$

i.e. $[x+1]-0$ if $-1 \le x < 0$.

Thus domain of f is $R-[-1,0)=\{x \notin [-1,0)\}$

We have $\sin\left(\frac{\pi}{[x+1]}\right)$ is continuous at all points of $R-[-1,0)$ and $[x]$ is continuous on $R-I,$ where I denotes the set of integers.

Thus the points where f can possibly be discontinuous are......, $-3,-2,-1,0\,1,2,..........$. But for $0 \le x < 1, [x]=0$ and $\sin\left(\frac{\pi}{[x+1]}\right)$ is defined.

Therefore $f(x)=0$ for $0 \le x < 1.$

Also $f(x)$ is not defined on $-1 \le x < 0.$

Therefore, continuity of f at 0 means continuity of f from right at 0. Since f is continuous from right at $0, f$ is continuous at 0. Hence set of points of discontinuities of f is $I-\{0\}$.

Example: 18 If the function $f(x)=\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x},(x \ne 0)$ is continuous at each point of its domain, then the value of $f(0)$ is

[Rajasthan PET 2000]

(a) 2 (b) 1/3 (c) 2/3 (d) – 1/3

Solution: (b) $f(x)=\lim_{x\to 0}\left(\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\right)=f(0) \quad ,\left(\frac{0}{0}\text{ form}\right)$

Applying L-Hospital's rule, $f(0)=\lim_{x\to 0}\frac{\left(2-\frac{1}{\sqrt{1-x^2}}\right)}{\left(2+\frac{1}{1+x^2}\right)}=\frac{2-1}{2+1}=\frac{1}{3}$

Trick : f (0) = $\lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x} \Rightarrow \lim_{x\to 0}\frac{2-\frac{\sin^{-1}x}{x}}{2+\frac{\tan^{-1}x}{x}}=\frac{2-1}{2+1}=\frac{1}{3}$.

Example: 19 The values of A and B such that the function $f(x)=\begin{cases}-2\sin x, & x \le -\frac{\pi}{2}\\ A\sin x+B, & -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \cos x, & x \ge \frac{\pi}{2}\end{cases}$, is continuous everywhere are

[Pb. CET 2000]

(a) $A=0, B=1$ (b) $A=1, B=1$ (c) $A=-1, B=1$ (d) $A=-1, B=0$

Solution: (c) For continuity at all $x \in R$, we must have $f\left(-\frac{\pi}{2}\right)=\lim_{x \to (-\pi/2)^-}(-2\sin x)=\lim_{x \to (-\pi/2)^+}(A\sin x+B)$

$\Rightarrow\ 2=-A+B$ .....(i)

and $f\left(\frac{\pi}{2}\right)=\lim_{x \to (\pi/2)^-}(A\sin x+B)=\lim_{x \to (\pi/2)^+}(\cos x)$

$\Rightarrow\ 0=A+B$ ....(ii)

From (i) and (ii), $A=-1$ and $B=1$.

Example: 20 If $f(x)=\frac{x^2-10x+25}{x^2-7x+10}$ for $x \neq 5$ and f is continuous at $x=5$, then $f(5)=$ **[EAMCET 2001]**

(a) 0 (b) 5 (c) 10 (d) 25

Solution: (a) $f(5)=\lim_{x \to 5} f(x)=\lim_{x \to 5}\frac{x^2-10x+25}{x^2-7x+10}=\lim_{x \to 5}\frac{(x-5)^2}{(x-2)(x-5)}=\frac{5-5}{5-2}=0$.

Example: 21 In order that the function $f(x)=(x+1)^{\cot x}$ is continuous at $x=0$, $f(0)$ must be defined as **[UPSEAT 2000; Haryana CEE 2001]**

(a) $f(0)=\frac{1}{e}$ (b) $f(0)=0$ (c) $f(0)=e$ (d) None of these

Solution: (c) For continuity at 0, we must have $f(0)=\lim_{x \to 0} f(x)$

$$=\lim_{x \to 0}(x+1)^{\cot x}=\lim_{x \to 0}\left\{(1+x)^{\frac{1}{x}}\right\}^{x\cot x}=\lim_{x \to 0}\left\{(1+x)^{\frac{1}{x}}\right\}^{\lim_{x \to 0}\left(\frac{x}{\tan x}\right)}=e^1=e.$$

Example: 22 The function $f(x)=\sin|x|$ is **[DCE 2002]**

(a) Continuous for all x (b) Continuous only at certain points

(c) Differentiable at all points (d) None of these

Solution: (a) It is obvious.

Example: 23 If $f(x)=\begin{cases}\frac{1-\sin x}{\pi-2x}, & x \neq \frac{\pi}{2} \\ \lambda, & x=\frac{\pi}{2}\end{cases}$ be continuous at $x=\frac{\pi}{2}$, then value of λ is **[Rajasthan PET 2002]**

(a) –1 (b) 1 (c) 0 (d) 2

Solution: (c) $f(x)$ is continuous at $x=\frac{\pi}{2}$, then $\lim_{x \to \pi/2} f(x)=f(0)$ or $\lambda=\lim_{x \to \pi/2}\frac{1-\sin x}{\pi-2x}$, $\left(\frac{0}{0}\text{ form}\right)$

Applying L-Hospital's rule, $\lambda=\lim_{x \to \pi/2}\frac{-\cos x}{-2} \Rightarrow \lambda=\lim_{x \to \pi/2}\frac{\cos x}{2}=0.$

Example: 24 If $f(x)=\frac{2-\sqrt{x+4}}{\sin 2x}$; $(x \neq 0)$, is continuous function at $x=0$, then $f(0)$ equals **[MP PET 2002]**

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

Solution: (d) If $f(x)$ is continuous at $x=0$, then, $f(0)=\lim_{x \to 0} f(x)=\lim_{x \to 0}\frac{2-\sqrt{x+4}}{\sin 2x}$, $\left(\frac{0}{0}\text{ form}\right)$

Using L–Hospital's rule, $f(0)=\lim_{x\to 0}\dfrac{\left(-\dfrac{1}{2\sqrt{x+4}}\right)}{2\cos 2x}=-\dfrac{1}{8}$.

Example: 25 If function $f(x)=\begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$, then $f(x)$ is continuous at ...... number of points

[UPSEAT 2002]

(a) ∞ (b) 1 (c) 0 (d) None of these

Solution: (c) At no point, function is continuous.

Example: 26 The function defined by $f(x)=\begin{cases} \left(x^2+e^{\frac{1}{2-x}}\right)^{-1} & , x\neq 2 \\ k & , x=2 \end{cases}$, is continuous from right at the point x = 2, then k is equal to

[Orissa JEE 2002]

(a) 0 (b) 1/4 (c) –1/4 (d) None of these

Solution: (b) $f(x)=\left[x^2+e^{\frac{1}{2-x}}\right]^{-1}$ and $f(2)=k$

If $f(x)$ is continuous from right at $x=2$ then $\lim_{x\to 2^+} f(x)=f(2)=k$

$\Rightarrow \lim_{x\to 2^+}\left[x^2+e^{\frac{1}{2-x}}\right]^{-1}=k \Rightarrow k=\lim_{h\to 0} f(2+h) \Rightarrow k=\lim_{h\to 0}\left[(2+h)^2+e^{\frac{1}{2-(2+h)}}\right]^{-1}$

$\Rightarrow k=\lim_{h\to 0}\left[4+h^2+4h+e^{-1/h}\right]^{-1} \Rightarrow k=[4+0+0+e^{-\infty}]^{-1} \Rightarrow k=\dfrac{1}{4}$.

Example: 27 The function $f(x)=\dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$ is not defined at $x=\pi$. The value of $f(\pi)$, so that $f(x)$ is continuous at $x=\pi$, is

[Orissa JEE 2003]

(a) $-\dfrac{1}{2}$ (b) $\dfrac{1}{2}$ (c) – 1 (d) 1

Solution: (c) $\lim_{x\to\pi} f(x)=\lim_{x\to\pi}\dfrac{2\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}=\lim_{x\to\pi}\dfrac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}=\lim_{x\to\pi}\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)$

$\therefore$ At $x=\pi$, $f(\pi)=-\tan\dfrac{\pi}{4}=-1$.

Example: 28 If $f(x)=\begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x} & ,\text{for } -1\leq x<0 \\ 2x^2+3x-2 & ,\text{for } 0\leq x\leq 1 \end{cases}$ is continuous at $x=0$, then $k=$ **[EAMCET 2003]**

(a) – 4 (b) – 3 (c) – 2 (d) – 1

Solution: (c) L.H.L. $=\lim_{x\to 0^-}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}=k$

R.H.L. $= \lim_{x \to 0^+} (2x^2 + 3x - 2) = -2$

Since it is continuous, hence L.H.L = R.H.L $\Rightarrow k = -2$.

Example: 29 The function $f(x) = |x| + \frac{|x|}{x}$ is **[Karnataka CET 2003]**

(a) Continuous at the origin

(b) Discontinuous at the origin because $|x|$ is discontinuous there

(c) Discontinuous at the origin because $\frac{|x|}{x}$ is discontinuous there

(d) Discontinuous at the origin because both $|x|$ and $\frac{|x|}{x}$ are discontinuous there

Solution: (c) $|x|$ is continuous at $x = 0$ and $\frac{|x|}{x}$ is discontinuous at $x = 0$

$\therefore f(x) = |x| + \frac{|x|}{x}$ is discontinuous at $x = 0$.