Chapter

6

Linear Programming

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*T**he linear programming originated during World War II, when the British and American military management called upon a group of Scientists to study and plan the war activities, so that maximum damages could be inflicted on the enemy camps at minimum cost and loss.*

*B**ecause of the success in military operations, it quickly spread in all phases of industry and government organizations. The Russian mathematician* L.Kantorovich *has, for a number of years, been interested in the applications of mathematics to programming problems. In 1941, F.L.* Hitchcock *formulated and solved the transportation problem. In 1947* T.C. Koopmans *solved the transportation problem. The minimum cost diet problem was studied by economist* Stigler *in 1945. In 1947,* G.B. Dantzig *suggested an efficient method known as the simplex method, which is an iterative procedure to solve any linear programming problem in a finite number of steps.*

Linear Programming

'Linear Programming' is a scientific tool to handle optimization problems. Here, we shall learn about some basic concepts of linear programming problems in two variables, their applications, advantages, limitations, formulation and graphical method of solution.
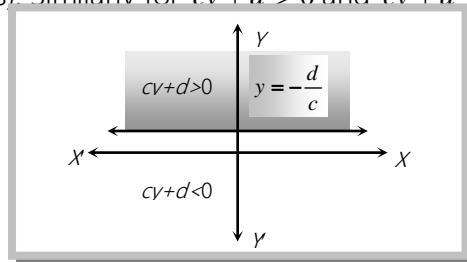
6.1 Linear Inequations

(1) Graph of linear inequations

(i) **Linear inequation in one variable :** $ax + b > 0, ax + b < 0, cy + d > 0$ etc. are called linear inequations in one variable. Graph of these inequations can be drawn as follows :
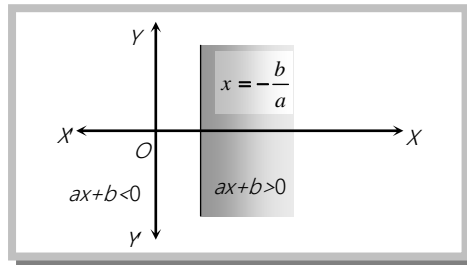

The graph of $ax + b > 0$ and $ax + b < 0$ are obtained by dividing xy-plane in two semi-planes by the line $x = -\dfrac{b}{a}$ (which is parallel to y-axis). Similarly for $cy + d > 0$ and $cy + d < 0$.

(ii) **Linear Inequation in two variables :** General form of these inequations are $ax + by > c, ax + by < c$. If any ordered pair (x_1, y_1) satisfies some inequations, then it is said to be a solution of the inequations.

The graph of these inequations is given below (for $c > 0$) :
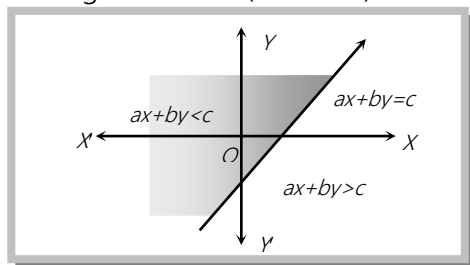
Working rule

To draw the graph of an inequation, following procedure is followed :

(i) Write the equation $ax + by = c$ in place of $ax + by < c$ and $ax + by > c$.

(ii) Make a table for the solutions of $ax + by = c$.

(iii) Now draw a line with the help of these points. This is the graph of the line $ax + by = c$.

(iv) If the inequation is > or <, then the points lying on this line is not considered and line is drawn dotted or discontinuous.

(v) If the inequation is $\geq$ or $\leq$, then the points lying on the line is considered and line is drawn bold or continuous.

(vi) This line divides the plane *XOY* in two region.

To Find the region that satisfies the inequation, we apply the following rules:

(a) Take an arbitrary point which will be in either region.

(b) If it satisfies the given inequation, then the required region will be the region in which the arbitrary point is located.

(c) If it does not satisfy the inequation, then the other region is the required region.

(d) Draw the lines in the required region or make it shaded.

(2) Simultaneous linear inequations in two variables : Since the solution set of a system of simultaneous linear inequations is the set of all points in two dimensional space which satisfy all the inequations simultaneously. Therefore to find the solution set we find the region of the plane common to all the portions

comprising the solution set of given inequations. In case there is no region common to all the solutions of the given inequations, we say that the solution set is void or empty.

(3) Feasible region : The limited (bounded) region of the graph made by two inequations is called feasible region. All the points in feasible region constitute the solution of a system of inequations. The feasible solution of a L.P.P. belongs to only first quadrant. If feasible region is empty then there is no solution for the problem.

Example: 1 Inequations $3x - y \geq 3$ and $4x - y > 4$

(a) Have solution for positive x and y
(b) Have no solution for positive x and y
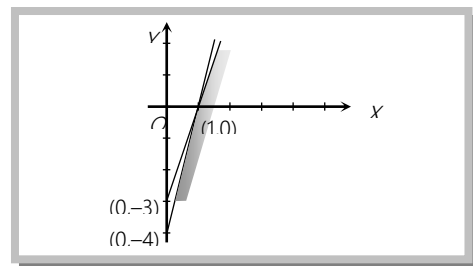(c) Have solution for all x
(d) Have solution for all y

Solution : (a) Following figure will be obtained on drawing the graphs of given inequations :

From $3x - y \geq 3, \dfrac{x}{1} + \dfrac{y}{-3} = 1$

From $4x - y \geq 4, \dfrac{x}{1} + \dfrac{y}{-4} = 1$

Clearly the common region of both the inequations is true for positive value of (x, y). It is also true for positive values of x and negative values of y.

Example: 2 The constraints   [MP PET 1999]

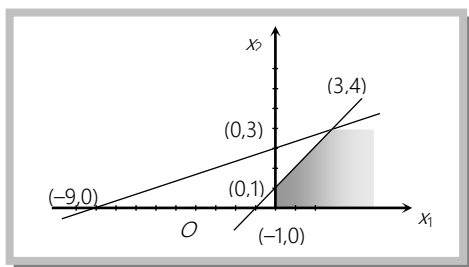$-x_1 + x_2 + \leq 1$; $-x_1 + 3x_2 \leq 9$; $x, x_2 \geq 0$ define

(a) Bounded feasible space
(b) Unbounded feasible space
(c) Both bounded and unbounded feasible space
(d) None of these

Solution : (b)
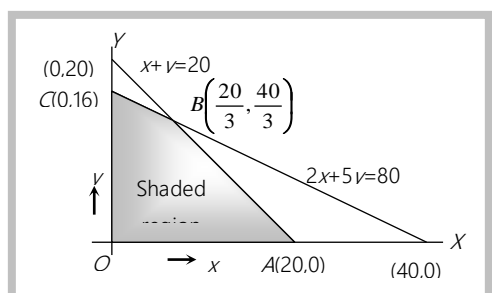

It is clear from the graph, the constraints define an unbounded feasible space.

Example: 3 Shaded region is represented by   [MP PET 1997]

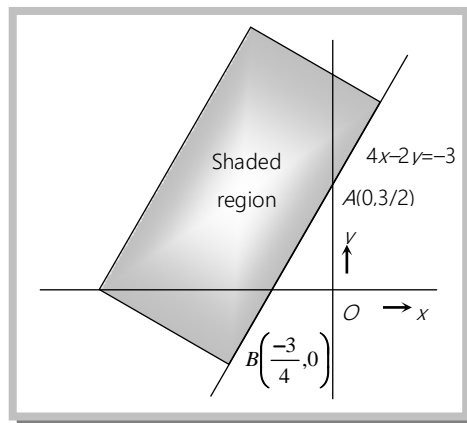(a) $2x + 5y \geq 80, x + y \leq 20, x \geq 0, y \leq 0$

(b) $2x + 5y \geq 80, x + y \geq 20, x \geq 0, y \geq 0$

(c) $2x + 5y \leq 80, x + y \leq 20, x \geq 0, y \geq 0$

(d) $2x + 5y \leq 80, x + y \leq 20, x \leq 0, y \leq 0$

Solution: (c)   In all the given equations, the origin is present in shaded area. Answer (c) satisfy this condition.

Example: 4   Shaded region is represented by

(a) $4x - 2y \leq 3$ (b) $4x - 2y \leq -3$ (c) $4x - 2y \geq 3$ (d) $4x - 2y \geq -3$

Solution: (b)   Origin is not present in given shaded area. So $4x - 2y \leq -3$ satisfy this condition.

6.2 Terms of Linear Programming

The term programming means planning and refers to a process of determining a particular program.

(1) Objective function : The linear function which is to be optimized (maximized or minimized) is called objective function of the *L.P.P.*

(2) Constraints or Restrictions : The conditions of the problem expressed as simultaneous equations or inequalities are called constraints or restrictions.

(3) **Non-negative Constraints :** Variables applied in the objective function of a linear programming problem are always non-negative. The ineqaulities which represent such constraints are called non-negative constraints.

(4) **Basic variables :** The m variables associated with columns of the $m \times n$ non-singular matrix which may be different from zero, are called basic variables.

(5) **Basic solution :** A solution in which the vectors associated to m variables are linear and the remaining $(n - m)$ variables are zero, is called a basic solution. A basic solution is called a degenerate basic solution, if at least one of the basic variables is zero and basic solution is called non-degenerate, if none of the basic variables is zero.

(6) **Feasible solution :** The set of values of the variables which satisfies the set of constraints of linear programming problem (*L.P.P*) is called a feasible solution of the *L.P.P.*

(7) **Optimal solution :** A feasible solution for which the objective function is minimum or maximum is called optimal solution.

(8) **Iso-profit line :** The line drawn in geometrical area of feasible region of *L.P.P.* for which the objective function remains constant at all the points lying on the line, is called iso-profit line.

If the objective function is to be minimized then these lines are called iso-cost lines.

(9) **Convex set :** In linear programming problems feasible solution is generally a polygon in first quadrant . This polygon is convex. It means if two points of polygon are connecting by a line, then the line must be inside to polygon. For example
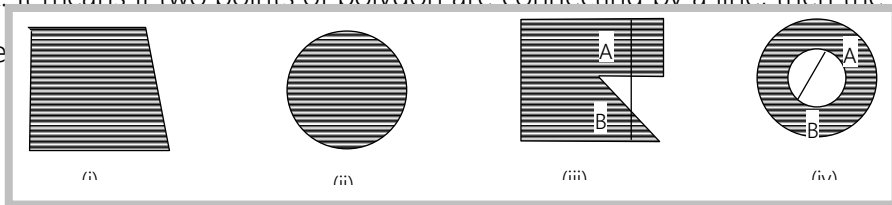

(i)　　　(ii)　　　(iii)　　　(iv)

Figure (i) and (ii) are convex set while (iii) and (iv) are not convex set

6.3 Mathemiatical formulation of a Linear programming problem

There are mainly four steps in the mathematical formulation of a linear programming problem, as mathematical model. We will discuss formulation of those problems which involve only two variables.

(1) Identify the decision variables and assign symbols x and y to them. These decision variables are those quantities whose values we wish to determine.

(2) Identify the set of constraints and express them as linear equations/inequations in terms of the decision variables. These constraints are the given conditions.

(3) Identify the objective function and express it as a linear function of decision variables. It might take the form of maximizing profit or production or minimizing cost.

(4) Add the non-negativity restrictions on the decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.

6.4 Graphical solution of two variable Linear programming problem

There are two techniques of solving an *L.P.P.* by graphical method. These are :

(1) Corner point method,   and   (2) Iso-profit or Iso-cost method.

(1) **Corner point method**

Working rule

(i) Formulate mathematically the *L.P.P.*

(ii) Draw graph for every constraint.

(iii) Find the feasible solution region.

(iv) Find the coordinates of the vertices of feasible solution region.

(v) Calculate the value of objective function at these vertices.

(vi) Optimal value (minimum or maximum) is the required solution.

Note : ❏ If there is no possibility to determine the point at which the suitable solution found, then the solution of problem is unbounded.

❏ If feasible region is empty, then there is no solution for the problem.

❏ Nearer to the origin, the objective function is minimum and that of further from the origin, the objective function is maximum.

(2) **Iso-profit or Iso-cost method :** Various steps of the method are as follows :

(i) Find the feasible region of the *L.P.P.*

(ii) Assign a constant value Z_1 to Z and draw the corresponding line of the objective function.

(iii) Assign another value Z_2 to Z and draw the corresponding line of the objective function.

(iv) If $Z_1 < Z_2, (Z_1 > Z_2)$, then in case of maximization (minimization) move the line P_1Q_1 corresponding to Z_1 to the line P_2Q_2 corresponding to Z_2 parallel to itself as far as possible, until the farthest point within the feasible region is touched by this line. The coordinates of the point give maximum (minimum) value of the objective function.

Note : ❏   The problem with more equations/inequations can be handled easily by this method.

❑ In case of unbounded region, it either finds an optimal solution or declares an unbounded solution. Unbounded solutions are not considered optimal solution. In real world problems, unlimited profit or loss is not possible.

6.5 To find the Vertices of Simple feasible region without Drawing a Graph

(1) **Bounded region:** The region surrounded by the inequalities $ax + by \leq m$ and $cx + dy \leq n$ in first quadrant is called bounded region. It is of the form of triangle or quadrilateral. Change these inequalities into equation, then by putting $x = 0$ and $y = 0,$ we get the solution also by solving the equation in which there may be the vertices of bounded region.

The maximum value of objective function lies at one vertex in limited region.

(2) **Unbounded region :** The region surrounded by the inequations $ax + by \geq m$ and $cx + dy \geq n$ in first quadrant, is called unbounded region.

Change the inequation in equations and solve for $x = 0$ and $y = 0$. Thus we get the vertices of feasible region.

The minimum value of objective function lies at one vertex in unbounded region but there is no existence of maximum value.
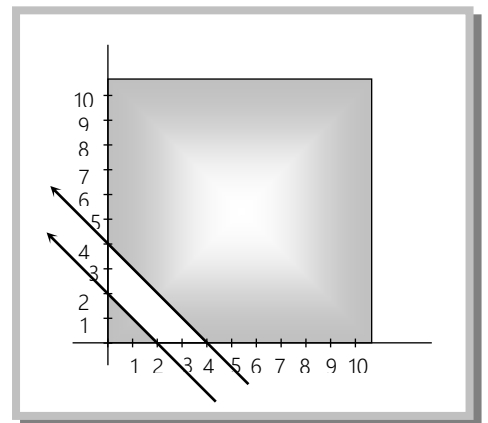
6.6 Problems having Infeasible Solutions

In some of the linear programming problems, constraints are inconsistent *i.e.* there does not exist any point which satisfies all the constraints. Such type of linear programming problems are said to have *infeasible solution*.

For Example : Maximize $Z = 5x + 2y$

subject to the constraints

$x + y \leq 2 , 3x + 3y \geq 2 , x, y \geq 0$

The above problem is illustrated graphically in the fig.

There is no point satisfying the set of above constraints. Thus, the problem is having an infeasible solution.

6.7 Some important points about L.P.P.

(1) If the constraints in a linear programming problem are changed, the problem is to be re-evaluated.

(2) The optimal value of the objective function is attained at the point, given by corner points of the feasible region.

(3) If a *L.P.P.* admits two optimal solutions, it has an infinite number of optimal solutions.

(4) If there is no possibility to determine the point at which the suitable solution can be found, then the solution of problem is unbounded.

(5) The maximum value of objective function lies at one vertex in limited region.

Example: 5 A firm makes pants and shirt. A shirt takes 2 hour on machine and 3 hour of man labour while a pant takes 3 hour on machine and 2 hour of man labour. In a week there are 70 hour of machine and 75 hour of man labour available. If the firm determines to make *x* shirts and *y* pants per week, then for this the linear constraints are

(a) $x \geq 0, y \geq 0, 2x + 3y \geq 70, 3x + 2y \geq 75$

(b) $x \geq 0, y \geq 0, 2x + 3y \leq 70, 3x + 2y \geq 75$

(c) $x \geq 0, y \geq 0, 2x + 3y \geq 70, 3x + 2y+ \leq 75$

(d) $x \geq 0, y \geq 0, 2x + 3y \leq 70, 3x + 2y \leq 75$

Solution: (d)

	Working time on machine	Man labour
Shirt (*x*)	2 hours	3 hours
Pant (*y*)	3 hours	2 hours
Availability	70 hours	75 hours

Linear constraints are $2x + 3y \leq 70, 3x + 2y \leq 75$ and $x, y \geq 0$

Example: 6 The minimum value of the objective function $Z = 2x + 10y$ for linear constraints

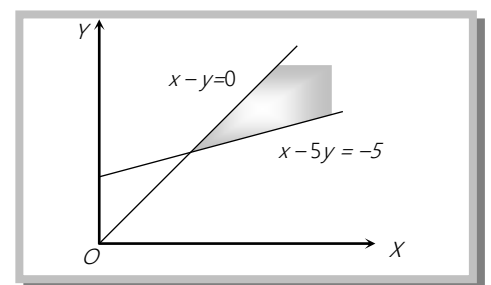$x - y \geq 0, x - 5y \leq -5$ and $x, y \geq 0$ is

(a) 10   (b) 15   (c) 12   (d) 8

Solution: (b) Required region is unbounded whose vertex is $\left(\dfrac{5}{4}, \dfrac{5}{4}\right)$

Hence the minimum value of objective function is $= 2 \times \dfrac{5}{4} + 10 \times \dfrac{5}{4} = 15$.

Example: 7 Minimize $z = \displaystyle\sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}x_{ij}$ **[MP PET 1999]**

Subject to : $\displaystyle\sum_{j=1}^{n} x_{ij} \leq a_i, i = 1........, m$; $\displaystyle\sum_{i=1}^{m} x_{ij} = b_j, j = 1......, n$ is a *LPP* with number of constraints

(a) $m + n$ (b) $m - n$ (c) mn (d) $\dfrac{m}{n}$

Solution: (a) (I) Condition, $i = 1, x_{11} + x_{12} + x_{13} + ... + x_{1n} \leq a_1$

$i = 2, x_{21} + x_{22} + x_{23} + ... + x_{2n} \leq a_2$

$$i = 3, x_{31} + x_{32} + x_{33} + \ldots \ldots .. + x_{3n} \leq a_3$$

....

$$i = m, x_{m1} + x_{m2} + x_{m3} + \ldots \ldots . + x_{mn} \leq a_m \rightarrow m \text{ constraint s}$$

(II) Condition $j = 1, x_{11} + x_{21} + x_{31} + \ldots \ldots + x_{m1} = b_1$

$$j = 2, x_{12} + x_{22} + x_{32} + \ldots \ldots . + x_{m2} = b_2$$

....

$$j = n, x_{1n} + x_{2n} + x_{3n} + \ldots \ldots \ldots + x_{mn} = b_n \rightarrow n \text{ constraints}$$

$\therefore$ Total constraints $= m + n$.

Example: 8   Maximize $z = 3x + 2y,$ subject to $x + y \geq 1, y - 5x \leq 0, x - y \geq -1, x + y \leq 6, x \leq 3$ and $x, y \geq 0$

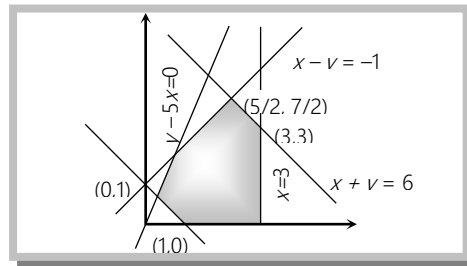(a) $x = 3$ (b) $y = 3$ (c) $z = 15$ (d) All the above

Solution: (d)   The shaded region represents the bounded region (3,3) satisfies, so $x = 3, y = 3$ and $z = 15$.

Example: 9   Maximum value of $4x + 5y$ subject to the constraints $x + y \leq 20, x + 2y \leq 35, x - 3y \leq 12$ is   **[Kurukshetra CEE 1996]**
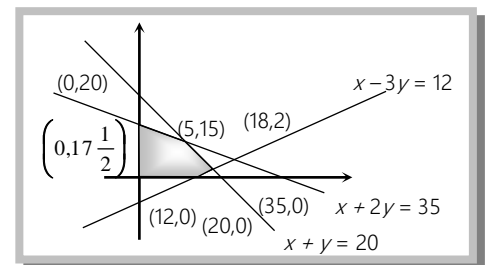
(a) 84   (b) 95   (c) 100   (d) 96

Solution: (b)   Obviously, *max.* $4x + 5y = 95$. It is at (5, 15).

Example: 10   The maximum value of $Z = 4x + 3y$ subjected to the constraints $3x + 2y \geq 160, \ 5x + 2y \geq 200, x + 2y \geq 80, x, y \geq 0$ is
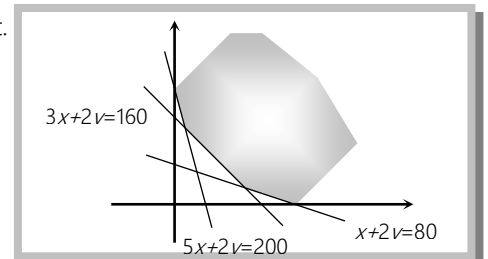
[MP PET 1998]

(a) 320   (b) 300   (c) 230   (d) None of these

Solution: (d)   Obviously, it is unbounded. Therefore its maximum value does not exist.
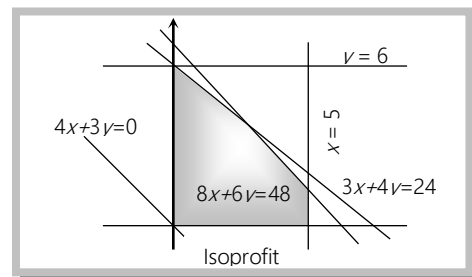
Example: 11   The objective function $Z = 4x + 3y$ can be maximized subjected to the constraints

$3x + 4y \leq 24, 8x + 6y \leq 48, x \leq 5, y \leq 6; x, y \geq 0$

(a) At only one point   (b) At two points only   (c) At an infinite number of points (d)   None of these

Solution: (c)   Obviously, the optimal solution is found on the line which is parallel to 'isoprofit line'.
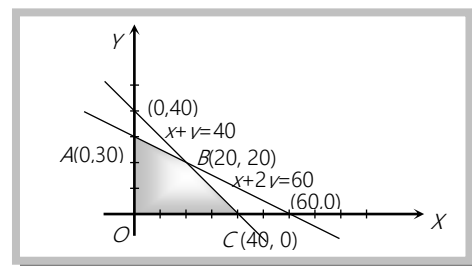
Hence it has infinite number of solutions.

Example: 12   The maximum value of $\mu = 3x + 4y$ subjected to the conditions $x + y \leq 40, x + 2y \leq 60; x, y \geq 0$ is   [MP PET 2002,04]

(a) 130   (b) 120   (c) 40   (d) 140

Solution: (d)   Obviously $Max \ \mu = 3x + 4y$ at $(20,20)$

$\mu = 60 + 80 = 140$.

6.8 Advantages and Limitations of L.P.P.

(1) **Advantages :** Linear programming is used to minimize the cost of production for maximum output. In short, with the help of linear programming models, a decision maker can most efficiently and effectively employ his production factor and limited resources to get maximum profit at minimum cost.

(2) **Limitations :** (i) The linear programming can be applied only when the objective function and all the constraints can be expressed in terms of linear equations/inequations.

(ii) Linear programming techniques provide solutions only when all the elements related to a problem can be quantified.

(iii) The coefficients in the objective function and in the constraints must be known with certainty and should remain unchanged during the period of study.

(iv) Linear programming technique may give fractional valued answer which is not desirable in some problems.