Chapter
3
Motion In Two Dimension

The motion of an object is called two dimensional, if two of the three co-ordinates required to specify the position of the object in space, change *w.r.t* time.

In such a motion, the object moves in a plane. For example, a billiard ball moving over the billiard table, an insect crawling over the floor of a room, earth revolving around the sun *etc*.

Two special cases of motion in two dimension are

1. Projectile motion   2. Circular motion

Introduction of Projectile Motion

A hunter aims his gun and fires a bullet directly towards a monkey sitting on a distant tree. If the monkey remains in his position, he will be safe but at the instant the bullet leaves the barrel of gun, if the monkey drops from the tree, the bullet will hit the monkey because the bullet will not follow the linear path.

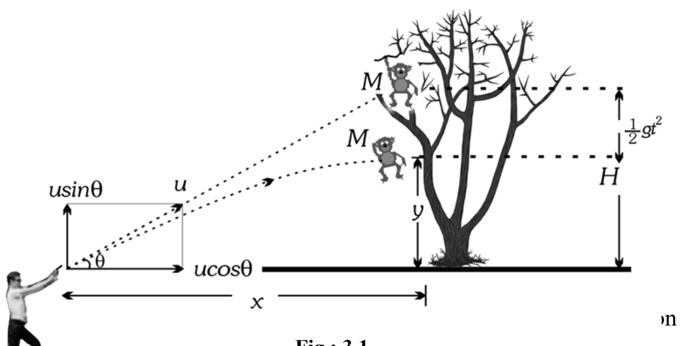


Fig : 3.1

...n of bullet is defined as projectile motion.

If the force acting on a particle is oblique with initial velocity then the motion of particle is called projectile motion.

Projectile

A body which is in flight through the atmosphere under the effect of gravity alone and is not being propelled by any fuel is called projectile.

Example:

(i) A bomb released from an aeroplane in level flight

(ii) A bullet fired from a gun

(iii) An arrow released from bow

(iv) A Javelin thrown by an athlete

Assumptions of Projectile Motion

(1) There is no resistance due to air.

(2) The effect due to curvature of earth is negligible.

(3) The effect due to rotation of earth is negligible.

(4) For all points of the trajectory, the acceleration due to gravity '*g*' is constant in magnitude and direction.

Principle of Physical Independence of Motions

(1) The motion of a projectile is a two-dimensional motion. So, it can be discussed in two parts. Horizontal motion and vertical motion. These two motions take place independent of each other. This is called the principle of physical independence of motions.

(2) The velocity of the particle can be resolved into two mutually perpendicular components. Horizontal component and vertical component.

(3) The horizontal component remains unchanged throughout the flight. The force of gravity continuously affects the vertical component.

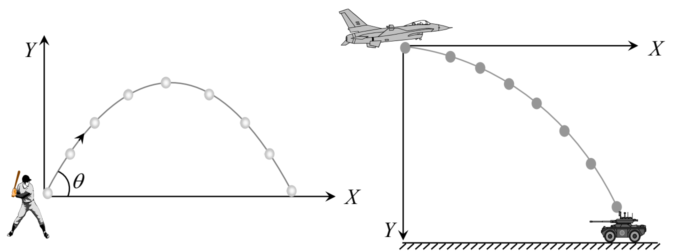
(4) The horizontal motion is a uniform motion and the vertical motion is a uniformly accelerated or retarded motion.

Types of Projectile Motion

(1) Oblique projectile motion

(2) Horizontal projectile motion

(3) Projectile motion on an inclined plane



Oblique Projectile

In projectile motion, horizontal component of velocity ($u\cos\theta$), acceleration (g) and mechanical energy remains constant while, speed, velocity, vertical component of velocity ($u\sin\theta$), momentum, kinetic energy and potential energy all changes. Velocity, and KE are maximum at the point of projection while minimum (but not zero) at highest point.

(1) **Equation of trajectory :** A projectile is thrown with velocity u at an angle θ with the horizontal. The velocity u can be resolved into two rectangular components.

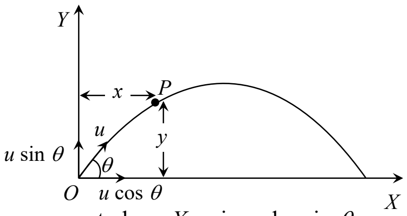


Fig : 3.3

$v\cos\theta$ component along X–axis and $u\sin\theta$ component along Y–axis.

For horizontal motion $x = u\cos\theta \times t \Rightarrow t = \dfrac{x}{u\cos\theta}$ … (i)

For vertical motion $y = (u\sin\theta)\,t - \dfrac{1}{2}gt^2$ …(ii)

From equation (i) and (ii)

$$y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x^2}{u^2\cos^2\theta}\right)$$

$$y = x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2\cos^2\theta}$$

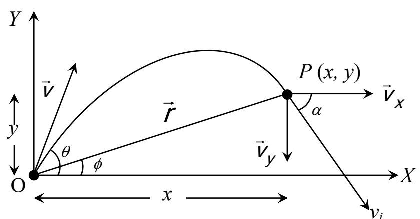
This equation shows that the trajectory of projectile is parabolic because it is similar to equation of parabola

$y = ax - bx^2$

Note : ❑ Equation of oblique projectile also can be written as

$$y = x\tan\theta\left[1 - \frac{x}{R}\right] \text{ (where } R = \text{horizontal range} = \frac{u^2\sin 2\theta}{g}\text{)}$$

(2) **Displacement of projectile $(\vec{r})$:** Let the particle acquires a position P having the coordinates (x, y) just after time t from the instant of projection. The corresponding position vector of the particle at time t is $\vec{r}$ as shown in the figure.



$\vec{r} = x\hat{i} + y\hat{j}$ …(i)

The horizontal distance covered during time t is given as

$x = v_x t \Rightarrow x = u\cos\theta\ t$ …(ii)

The vertical velocity of the particle at time t is given as

$v_y = (v_0)_y - gt,$ …(iii)

Now the vertical displacement y is given as

$y = u\sin\theta\, t - 1/2\, gt^2$ …(iv)

Putting the values of x and y from equation (ii) and equation (iv) in equation (i) we obtain the position vector at any time t as

$$\vec{r} = (u\cos\theta)\,t\hat{i} + \left((u\sin\theta)\,t - \frac{1}{2}gt^2\right)\hat{j}$$

$$\Rightarrow r = \sqrt{(ut\cos\theta)^2 + \left((ut\sin\theta) - \frac{1}{2}gt^2\right)^2}$$

$$r = ut\sqrt{1 + \left(\frac{gt}{2u}\right)^2 - \frac{gt\sin\theta}{u}} \text{ and } \phi = \tan^{-1}(y/x)$$

$$= \tan^{-1}\left(\frac{ut\sin\theta - \frac{1}{2}gt^2}{(ut\cos\theta)}\right) \text{ or } \phi = \tan^{-1}\left(\frac{2u\sin\theta - gt}{2u\cos\theta}\right)$$

Note : ❑ The angle of elevation ϕ of the highest point of the projectile and the angle of projection θ are related to each other as

$$\tan\phi = \frac{1}{2}\tan\theta$$

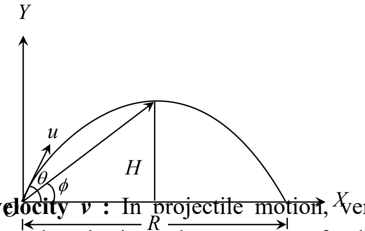


Fig : 3.5

(3) **Instantaneous velocity v :** In projectile motion, vertical component of velocity changes but horizontal component of velocity remains always constant.

Example : When a man jumps over the hurdle leaving behind its skateboard then vertical component of his velocity is changing, but not the horizontal component which matches with the skateboard velocity.

As a result, the skateboard stays underneath him, allowing him to land on it.

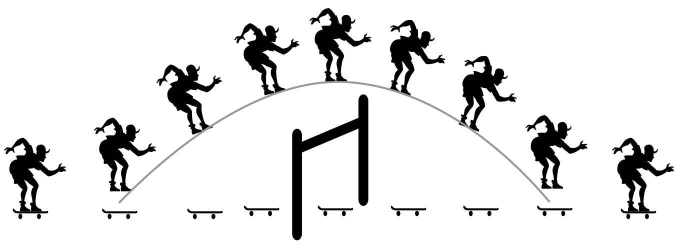


Fig : 3.6

Let v_i be the instantaneous velocity of projectile at time t, direction of this velocity is along the tangent to the trajectory at point P.

$$\vec{v}_i = v_x \hat{i} + v_y \hat{j} \Rightarrow v_i = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{u^2 \cos^2\theta + (u\sin\theta - gt)^2}$$

$$v_i = \sqrt{u^2 + g^2t^2 - 2ugt\sin\theta}$$

Direction of instantaneous velocity $\tan\alpha = \dfrac{v_y}{v_x} = \dfrac{u\sin\theta - gt}{u\cos\theta}$

or $\alpha = \tan^{-1}\left[\tan\theta - \dfrac{gt}{u}\sec\theta\right]$

(4) **Change in velocity :** Initial velocity (at projection point) $\vec{u}_i = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$

Final velocity (at highest point) $\vec{u}_f = u\cos\theta\,\hat{i} + 0\,\hat{j}$

(i) Change in velocity (Between projection point and highest point) $\Delta\vec{u} = \vec{u}_f - \vec{u}_i = -u\sin\theta\,\hat{j}$

When body reaches the ground after completing its motion then final velocity $\vec{u}_f = u\cos\theta\,\hat{i} - u\sin\theta\,\hat{j}$

(ii) Change in velocity (Between complete projectile motion) $\Delta\vec{u} = u_f - u_i = -2u\sin\theta\,\hat{i}$

(5) **Change in momentum :** Simply by the multiplication of mass in the above expression of velocity (Article-4).

(i) Change in momentum (Between projection point and highest point) $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = -mu\sin\theta\,\hat{j}$

(ii) Change in momentum (For the complete projectile motion) $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = -2mu\sin\theta\,\hat{j}$

(6) **Angular momentum :** Angular momentum of projectile at highest point of trajectory about the point of projection is given by

$$L = mvr \qquad \left[\text{Here } r = H = \frac{u^2\sin^2\theta}{2g}\right]$$

$$\therefore \quad L = m\,u\cos\theta\,\frac{u^2\sin^2\theta}{2g} = \frac{m\,u^3\cos\theta\sin^2\theta}{2g}$$

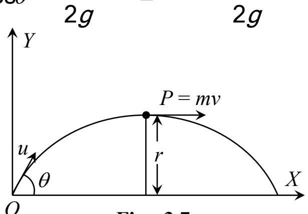


Fig : 3.7

(7) **Time of flight :** The total time taken by the projectile to go up and come down to the same level from which it was projected is called time of flight.

For vertical upward motion $0 = u\sin\theta - gt$

$\Rightarrow t = (u\sin\theta/g)$

Now as time taken to go up is equal to the time taken to come down so

Time of flight $T = 2t = \dfrac{2u\sin\theta}{g}$

(i) Time of flight can also be expressed as : $T = \dfrac{2.u_y}{g}$ (where u_y is the vertical component of initial velocity).

(ii) For complementary angles of projection θ and $90^\circ - \theta$

(a) Ratio of time of flight $= \dfrac{T_1}{T_2} = \dfrac{2u\sin\theta/g}{2u\sin(90-\theta)/g}$

$= \tan\theta \Rightarrow \dfrac{T_1}{T_2} = \tan\theta$

(b) Multiplication of time of flight $= T_1T_2 = \dfrac{2u\sin\theta}{g}\dfrac{2u\cos\theta}{g}$

$\Rightarrow T_1T_2 = \dfrac{2R}{g}$

(iii) If t_1 is the time taken by projectile to rise upto point p and t_2 is the time taken in falling from point p to ground level then $t_1 + t_2 = \dfrac{2u\sin\theta}{g} =$ time of flight or $u\sin\theta = \dfrac{g(t_1+t_2)}{2}$

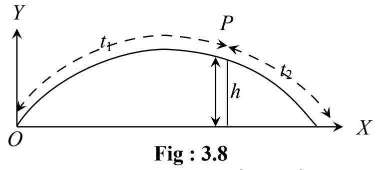


Fig : 3.8

and height of the point p is given by $h = u\sin\theta\, t_1 - \dfrac{1}{2}gt_1^2$

$$h = g\frac{(t_1+t_2)}{2}t_1 - \frac{1}{2}gt_1^2$$

by solving $h = \dfrac{gt_1t_2}{2}$

(iv) If B and C are at the same level on trajectory and the time difference between these two points is t_1, similarly A and D are also at the same level and the time difference between these two positions is t_2 then

$$t_2^2 - t_1^2 = \frac{8h}{g}$$

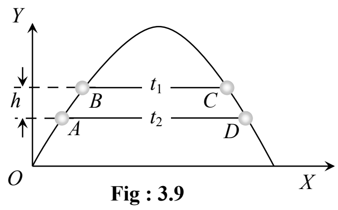


Fig : 3.9

(8) **Horizontal range :** It is the horizontal distance travelled by a body during the time of flight.

So by using second equation of motion in x-direction

$R = u\cos\theta \times T$

$= u\cos\theta \times (2u\sin\theta/g)$

$= \dfrac{u^2\sin 2\theta}{g}$

$$R = \frac{u^2\sin 2\theta}{g}$$

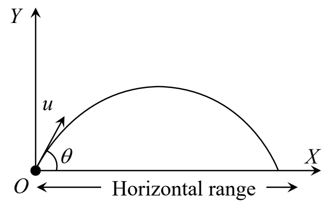


Fig : 3.10

(i) Range of projectile can also be expressed as :

$R = u\cos\theta \times T = u\cos\theta \dfrac{2u\sin\theta}{g}$

$= \dfrac{2u\cos\theta\ u\sin\theta}{g} = \dfrac{2u_x u_y}{g}$

$\therefore\ R = \dfrac{2u_x u_y}{g}$ (where u_x and u_y are the horizontal and vertical component of initial velocity)

(ii) If angle of projection is changed from θ to $\theta' = (90 - \theta)$ then range remains unchanged.

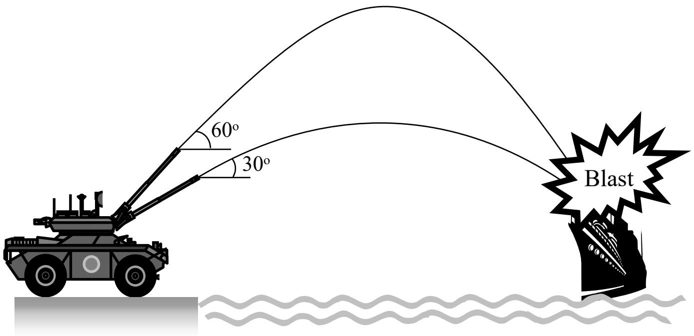


Fig : 3.11

$$R = \frac{u^2\sin 2\theta'}{g} = \frac{u^2\sin[2(90^o - \theta)]}{g} = \frac{u^2\sin 2\theta}{g} = R$$

So a projectile has same range at angles of projection θ and $(90 - \theta)$, though time of flight, maximum height and trajectories are different.

These angles θ and $90^o - \theta$ are called complementary angles of projection and for complementary angles of projection, ratio of range

$$\frac{R_1}{R_2} = \frac{u^2\sin 2\theta / g}{u^2\sin[2(90^o - \theta)]/g} = 1 \Rightarrow \frac{R_1}{R_2} = 1$$

(iii) For angle of projection $\theta_1 = (45 - \alpha)$ and $\theta_2 = (45 + \alpha)$, range will be same and equal to $u^2\cos 2\alpha/g$.

θ_1 and θ_2 are also the complementary angles.

(iv) Maximum range : For range to be maximum

$$\frac{dR}{d\theta} = 0 \Rightarrow \frac{d}{d\theta}\left[\frac{u^2\sin 2\theta}{g}\right] = 0$$

$\Rightarrow \cos 2\theta = 0$ *i.e.* $2\theta = 90^o \Rightarrow \theta = 45^o$

and $R_{max} = (u^2/g)$

i.e., a projectile will have maximum range when it is projected at an angle of 45° to the horizontal and the maximum range will be (u^2/g).

When the range is maximum, the height H reached by the projectile

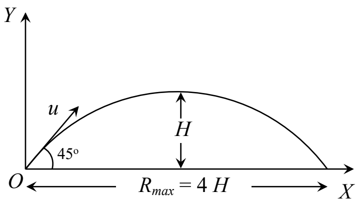


Fig : 3.12

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin^2 45}{2g} = \frac{u^2}{4g} = \frac{R_{max}}{4}$$

i.e., if a person can throw a projectile to a maximum distance R_{max}, The maximum height during the flight to which it will rise is $\left(\dfrac{R_{max}}{4}\right)$.

(v) Relation between horizontal range and maximum height : $R = \dfrac{u^2\sin 2\theta}{g}$ and $H = \dfrac{u^2\sin^2\theta}{2g}$

$$\therefore\ \frac{R}{H} = \frac{u^2\sin 2\theta / g}{u^2\sin^2\theta / 2g} = 4\cot\theta \Rightarrow R = 4H\cot\theta$$

(vi) If in case of projectile motion range R is n times the maximum height H

i.e. $R = nH \Rightarrow \dfrac{u^2\sin 2\theta}{g} = n\dfrac{u^2\sin^2\theta}{2g}$

$\Rightarrow \tan\theta = [4/n]$ or $\theta = \tan^{-1}[4/n]$

The angle of projection is given by $\theta = \tan^{-1}[4/n]$

Note : ❑ If $R = H$ then $\theta = \tan^{-1}(4)$ or $\theta = 76^o$.

If $R = 4H$ then $\theta = \tan^{-1}(1)$ or $\theta = 45^o$.

(9) **Maximum height :** It is the maximum height from the point of projection, a projectile can reach.

So, by using $v^2 = u^2 + 2as$

$0 = (u\sin\theta)^2 - 2gH$

$H = \dfrac{u^2\sin^2\theta}{2g}$

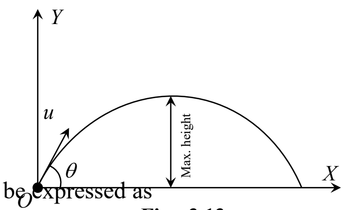


Fig : 3.13

(i) Maximum height can also be expressed as

$H = \dfrac{u_y^2}{2g}$ (where u_y is the vertical component of initial velocity).

(ii) $H_{max} = \dfrac{u^2}{2g}$ (when $\sin^2\theta = \max = 1$ *i.e.,* $\theta = 90^o$)

i.e., for maximum height body should be projected vertically upward. So it falls back to the point of projection after reaching the maximum height.

(iii) For complementary angles of projection θ and $90^o - \theta$

Ratio of maximum height

$$= \frac{H_1}{H_2} = \frac{u^2\sin^2\theta / 2g}{u^2\sin^2(90^o - \theta)/2g} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

$$\therefore\ \frac{H_1}{H_2} = \tan^2\theta$$

(10) **Projectile passing through two different points on same height at time t_1 and t_2 :** If the particle passes two points situated at equal height y at $t = t_1$ and $t = t_2$, then

(i) **Height (y):** $y = (u\sin\theta)t_1 - \frac{1}{2}gt_1^2$ …(i)

and $y = (u\sin\theta)t_2 - \frac{1}{2}gt_2^2$ …(ii)

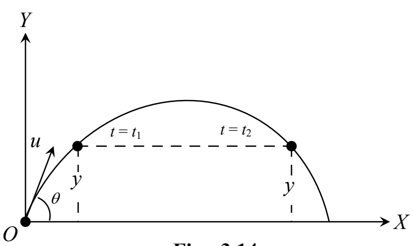


Fig : 3.14

Comparing equation (i) with equation (ii)

$$u\sin\theta = \frac{g(t_1 + t_2)}{2}$$

Substituting this value in equation (i)

$$y = g\left(\frac{t_1 + t_2}{2}\right)t_1 - \frac{1}{2}gt_1^2 \Rightarrow y = \frac{gt_1 t_2}{2}$$

(ii) **Time (t_1 and t_2):** $y = u\sin\theta\, t - \frac{1}{2}gt^2$

$$t^2 - \frac{2u\sin\theta}{g}t + \frac{2y}{g} = 0 \Rightarrow t = \frac{u\sin\theta}{g}\left[1 \pm \sqrt{1 - \left(\frac{\sqrt{2gy}}{u\sin\theta}\right)^2}\right]$$

$$t_1 = \frac{u\sin\theta}{g}\left[1 + \sqrt{1 - \left(\frac{\sqrt{2gy}}{u\sin\theta}\right)^2}\right]$$

and $t_2 = \frac{u\sin\theta}{g}\left[1 - \sqrt{1 - \left(\frac{\sqrt{2gy}}{u\sin\theta}\right)^2}\right]$

(11) **Motion of a projectile as observed from another projectile :** Suppose two balls *A* and *B* are projected simultaneously from the origin, with initial velocities u_1 and u_2 at angle θ_1 and θ_2, respectively with the horizontal.

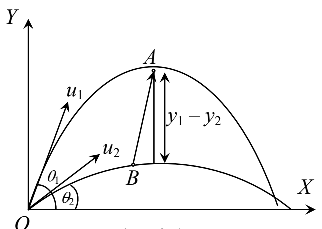


Fig : 3.15

The instantaneous positions of the two balls are given by

Ball *A* : $x_1 = (u_1\cos\theta_1)t$, $y_1 = (u_1\sin\theta_1)t - \frac{1}{2}gt^2$

Ball *B* : $x_2 = (u_2\cos\theta_2)t$, $y_2 = (u_2\sin\theta_2)t - \frac{1}{2}gt^2$

The position of the ball *A* with respect to ball *B* is given by

$$x = x_1 - x_2 = (u_1\cos\theta_1 - u_2\cos\theta_2)t$$

$$y = y_1 - y_2 = (u_1\sin\theta_1 - u_2\sin\theta_2)t$$

Now $\frac{y}{x} = \left(\frac{u_1\sin\theta_1 - u_2\sin\theta_2}{u_1\cos\theta_1 - u_2\cos\theta_2}\right) =$ constant

Thus motion of a projectile relative to another projectile is a straight line.

(12) **Energy of projectile :** When a projectile moves upward its kinetic energy decreases, potential energy increases but the total energy always remain constant.

If a body is projected with initial kinetic energy $K(=1/2\ mu^2)$, with angle of projection θ with the horizontal then at the highest point of trajectory

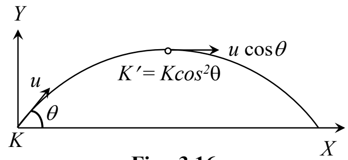


Fig : 3.16

(i) **Kinetic energy** $= \frac{1}{2}m(u\cos\theta)^2 = \frac{1}{2}mu^2\cos^2\theta$

$\therefore\ K' = K\cos^2\theta$

(ii) **Potential energy** $= mgH = mg\frac{u^2\sin^2\theta}{2g}$

$= \frac{1}{2}mu^2\sin^2\theta$ $\left(\text{As } H = \frac{u^2\sin^2\theta}{2g}\right)$

$= K\sin^2\theta$

(iii) **Total energy** = Kinetic energy + Potential energy

$= \frac{1}{2}mu^2\cos^2\theta + \frac{1}{2}mu^2\sin^2\theta$

$= \frac{1}{2}mu^2 =$ Energy at the point of projection.

This is in accordance with the law of conservation of energy.

Horizontal Projectile

When a body is projected horizontally from a certain height '*y*' vertically above the ground with initial velocity *u*. If friction is considered to be absent, then there is no other horizontal force which can affect the horizontal motion. The horizontal velocity therefore remains constant and so the object covers equal distance in horizontal direction in equal intervals of time.

The horizontal velocity therefore remains constant and so the object covers equal distance in horizontal direction in equal intervals of time.

(1) **Trajectory of horizontal projectile :** The horizontal displacement *x* is governed by the equation

$x = ut \Rightarrow t = \frac{x}{u}$ …. (i)

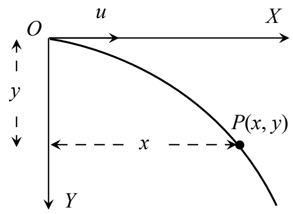


Fig : 3.17

The vertical displacement *y* is governed by

$y = \frac{1}{2}gt^2$ …. (ii)

(since initial vertical velocity is zero)

By substituting the value of *t* in equation (ii) $y = \frac{1}{2}\frac{gx^2}{u^2}$

(2) **Displacement of Projectile** $(\vec{r})$: After time *t*, horizontal displacement $x = ut$ and vertical displacement $y = \frac{1}{2}gt^2$.

So, the position vector $\vec{r} = ut\hat{i} + \frac{1}{2}gt^2\hat{j}$

Therefore $r = ut\sqrt{1+\left(\frac{gt}{2u}\right)^2}$ and $\alpha = \tan^{-1}\left(\frac{gt}{2u}\right)$

$$\alpha = \tan^{-1}\left(\sqrt{\frac{gy}{2}}\Big/u\right) \qquad \left(\text{as } t = \sqrt{\frac{2y}{g}}\right)$$

(3) **Instantaneous velocity :** Throughout the motion, the horizontal component of the velocity is $v_x = u$.

The vertical component of velocity increases with time and is given by

$v_y = 0 + g\,t = g\,t$ (From $v = u + g\,t$)

So, $\vec{v} = v_x\hat{i} + v_y\hat{j} = u\hat{i} + gt\hat{j}$

i.e. $v = \sqrt{u^2 + (gt)^2} = u\sqrt{1+\left(\frac{gt}{u}\right)^2}$

Again $\vec{v} = u\hat{i} + \sqrt{2gy}\,\hat{j}$

i.e. $v = \sqrt{u^2 + 2gy}$

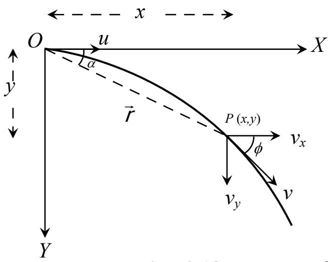


Fig : 3.18

Direction of instantaneous velocity : $\tan\phi = \frac{v_y}{v_x}$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{\sqrt{2gy}}{u}\right) \text{ or } \phi = \tan^{-1}\left(\frac{gt}{u}\right)$$

Where ϕ is the angle of instantaneous velocity from the horizontal.

(4) **Time of flight :** If a body is projected horizontally from a height h with velocity u and time taken by the body to reach the ground is T, then

$h = 0 + \frac{1}{2}gT^2$ (for vertical motion)

$$T = \sqrt{\frac{2h}{g}}$$

(5) **Horizontal range :** Let R is the horizontal distance travelled by the body

$R = uT + \frac{1}{2}\,0\,T^2$ (for horizontal motion)

$$R = u\sqrt{\frac{2h}{g}}$$

(6) If projectiles A and B are projected horizontally with different initial velocity from same height and third particle C is dropped from same point then

(i) All three particles will take equal time to reach the ground.

(ii) Their net velocity would be different but all three particle possess same vertical component of velocity.

(iii) The trajectory of projectiles A and B will be straight line *w.r.t.* particle C.

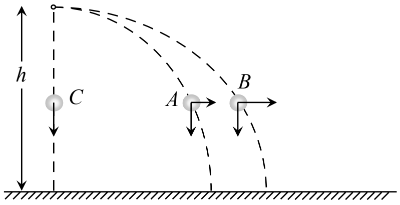


Fig : 3.19

(7) If various particles thrown with same initial velocity but in different direction then

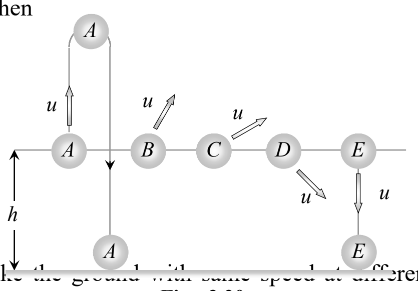


Fig : 3.20

(i) They strike the ground with same speed at different times irrespective of their initial direction of velocities.

(ii) Time would be least for particle E which was thrown vertically downward.

(iii) Time would be maximum for particle A which was thrown vertically upward.

Projectile Motion on An Inclined Plane

Let a particle be projected up with a speed u from an inclined plane which makes an angle α with the horizontal and velocity of projection makes an angle θ with the inclined plane.

We have taken reference x-axis in the direction of plane.

Hence the component of initial velocity parallel and perpendicular to the plane are equal to $u\cos\theta$ and $u\sin\theta$ respectively *i.e.* $u_{\parallel} = u\cos\theta$ and $u_{\perp} = u\sin\theta$.

The component of g along the plane is $g\sin\alpha$ and perpendicular to the plane is $g\cos\alpha$ as shown in the figure *i.e.* $a_{\parallel} = -g\sin\alpha$ and $a_{\perp} = g\cos\alpha$.

Therefore the particle decelerates at a rate of $g\sin\alpha$ as it moves from O to P.

(1) **Time of flight :** We know for oblique projectile motion $T = \frac{2u\sin\theta}{g}$ or we can say $T = \frac{2u_{\perp}}{a_{\perp}}$

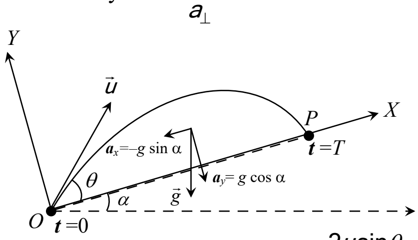


Fig : 3.21

$\therefore$ Time of flight on an inclined plane $T = \frac{2u\sin\theta}{g\cos\alpha}$

(2) **Maximum height :** We know for oblique projectile motion $H = \frac{u^2\sin^2\theta}{2g}$ or we can say $H = \frac{u_{\perp}^2}{2a_{\perp}}$

$\therefore$ Maximum height on an inclined plane $H = \frac{u^2\sin^2\theta}{2g\cos\alpha}$

(3) **Horizontal range :** For one dimensional motion $s = ut + \frac{1}{2}at^2$

Horizontal range on an inclined plane $R = u_{\|}\,T + \frac{1}{2}a_{\|}\,T^2$

$$R = u\cos\theta\,T - \frac{1}{2}g\sin\alpha\,T^2$$

$$R = u\cos\theta\left(\frac{2u\sin\theta}{g\cos\alpha}\right) - \frac{1}{2}g\sin\alpha\left(\frac{2u\sin\theta}{g\cos\alpha}\right)^2$$

By solving $R = \frac{2u^2}{g}\frac{\sin\theta\cos(\theta+\alpha)}{\cos^2\alpha}$

(i) Maximum range occurs when $\theta = \frac{\pi}{4} - \frac{\alpha}{2}$

(ii) The maximum range along the inclined plane when the projectile is thrown upwards is given by

$$R_{\max} = \frac{u^2}{g(1+\sin\alpha)}$$

(iii) The maximum range along the inclined plane when the projectile is thrown downwards is given by

$$R_{\max} = \frac{u^2}{g(1-\sin\alpha)}$$

Circular Motion

Circular motion is another example of motion in two dimensions. To create circular motion in a body it must be given some initial velocity and a force must then act on the body which is always directed at right angles to instantaneous velocity.

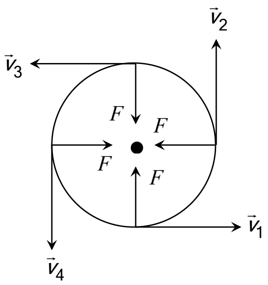


Fig : 3.22

Since this force is always at right angles to the displacement therefore no work is done by the force on the particle. Hence, its kinetic energy and thus speed is unaffected. But due to simultaneous action of the force and the velocity the particle follows resultant path, which in this case is a circle. Circular motion can be classified into two types – Uniform circular motion and non-uniform circular motion.

Variables of Circular Motion

(1) **Displacement and distance :** When particle moves in a circular path describing an angle θ during time t (as shown in the figure) from the position A to the position B, we see that the magnitude of the position vector $\vec{r}$ (that is equal to the radius of the circle) remains constant. *i.e.*, $|\vec{r}_1| = |\vec{r}_2| = r$ and the direction of the position vector changes from time to time.

(i) Displacement **:** The change of position vector or the displacement $\Delta\vec{r}$ of the particle from position A to the position B is given by referring the figure.

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 \Rightarrow \Delta r = |\Delta\vec{r}| = |\vec{r}_2 - \vec{r}_1|$$

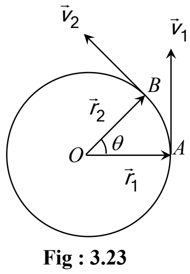


Fig : 3.23

$$\Delta r = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}$$

Putting $r_1 = r_2 = r$ we obtain

$$\Delta r = \sqrt{r^2 + r^2 - 2r.r\cos\theta}$$

$$\Rightarrow \Delta r = \sqrt{2r^2(1-\cos\theta)}$$

$$= \sqrt{2r^2\left(2\sin^2\frac{\theta}{2}\right)}$$

$$\Delta r = 2r\sin\frac{\theta}{2}$$

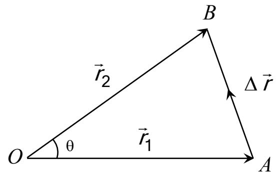


Fig : 3.24

(ii) Distance **:** The distanced covered by the particle during the time t is given as

d = length of the arc $AB = r\ \theta$

(iii) Ratio of distance and displacement **:** $\frac{d}{\Delta r} = \frac{r\theta}{2r\sin\theta/2}$

$$= \frac{\theta}{2}\operatorname{cosec}(\theta/2)$$

(2) **Angular displacement (θ) :** The angle turned by a body moving in a circle from some reference line is called angular displacement.

(i) Dimension = $[M^0L^0T^0]$ (as θ = arc/ radius).

(ii) Units = Radian or Degree. It is some time also specified in terms of fraction or multiple of revolution.

(iii) $2\pi\,\text{rad} = 360^o = 1\,\text{Revolution}$

(iv) Angular displacement is a axial vector quantity.

Its direction depends upon the sense of rotation of the object and can be given by Right Hand Rule; which states that if the curvature of the fingers of right hand represents the sense of rotation of the object, then the thumb, held perpendicular to the curvature of the fingers, represents the direction of angular displacement vector.

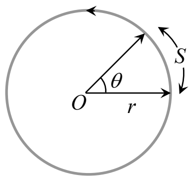


Fig : 3.25

(v) Relation between linear displacement and angular displacement $\vec{s} = \vec{\theta} \times \vec{r}$

or $s = r\theta$

(3) **Angular velocity (ω) :** Angular velocity of an object in circular motion is defined as the time rate of change of its angular displacement.

(i) Angular velocity $\omega = \frac{\text{angle traced}}{\text{time taken}} = \underset{\Delta t \to 0}{Lt}\frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

$$\therefore\ \omega = \frac{d\theta}{dt}$$

(ii) Dimension : $[M^0L^0T^{-1}]$

(iii) Units : Radians per second ($rad.s^{-1}$) or Degree per second.

(iv) Angular velocity is an axial vector.

Its direction is the same as that of $\Delta\theta$. For anticlockwise rotation of the point object on the circular path, the direction of ω, according to Right hand rule is along the axis of circular path directed upwards. For clockwise rotation of the point object on the circular path, the direction of ω is along the axis of circular path directed downwards.

(v) Relation between angular velocity and linear velocity $\vec{v} = \vec{\omega} \times \vec{r}$

(vi) For uniform circular motion ω remains constant where as for non-uniform motion ω varies with respect to time.

Note : ❑ It is important to note that nothing actually moves in the direction of the angular velocity vector $\vec{\omega}$. The direction of $\vec{\omega}$ simply represents that the circular motion is taking place in a plane perpendicular to it.

(4) **Change in velocity :** We want to know the magnitude and direction of the change in velocity of the particle which is performing uniform circular motion as it moves from A to B during time t as shown in figure. The change in velocity vector is given as

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$$

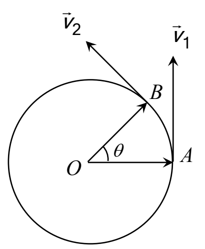


Fig : 3.26

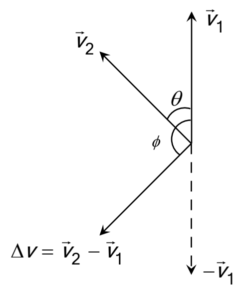


Fig : 3.27

or $|\Delta\vec{v}| = |\vec{v}_2 - \vec{v}_1| \Rightarrow \Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\theta}$

For uniform circular motion $v_1 = v_2 = v$

So $\Delta v = \sqrt{2v^2(1-\cos\theta)} = 2v\sin\frac{\theta}{2}$

The direction of $\Delta\vec{v}$ is shown in figure that can be given as

$$\phi = \frac{180^o - \theta}{2} = (90^o - \theta/2)$$

(5) **Time period (*T*) :** In circular motion, the time period is defined as the time taken by the object to complete one revolution on its circular path.

(i) Units : second.

(ii) Dimension : $[M^0L^0T]$

(iii) Time period of second's hand of watch = 60 *second.*

(iv) Time period of minute's hand of watch = 60 *minute*

(v) Time period of hour's hand of watch = 12 *hour*

(6) **Frequency (*n*) :** In circular motion, the frequency is defined as the number of revolutions completed by the object on its circular path in a unit time.

(i) Units : s^{-1} or hertz (*Hz*).

(ii) Dimension : $[M^0L^0T^{-1}]$

Note : ❑ Relation between time period and frequency : If n is the frequency of revolution of an object in circular motion, then the object completes n revolutions in 1 second. Therefore, the object will complete one revolution in $1/n$ second.

$\therefore T = 1/n$

❑ Relation between angular velocity, frequency and time period : Consider a point object describing a uniform circular motion with frequency n and time period T. When the object completes one revolution, the angle traced at its axis of circular motion is 2π radians. It means, when time $t = T$, $\theta = 2\pi$ radians. Hence, angular velocity

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi n \quad (\because T = 1/n)$$

$$\omega = \frac{2\pi}{T} = 2\pi n$$

❑ If two particles are moving on same circle or different coplanar concentric circles in same direction with different uniform angular speeds ω_A and ω_B respectively, the angular velocity of B relative to A will be

$$\omega_{rel} = \omega_B - \omega_A$$

So the time taken by one to complete one revolution around O with respect to the other (i.e., time in which B complete one revolution around O with respect to the other (*i.e.*, time in which B completes one more or less revolution around O than A)

$$T = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{\omega_2 - \omega_1} = \frac{T_1 T_2}{T_1 - T_2} \qquad \left[\text{as } T = \frac{2\pi}{\omega}\right]$$

Special case : If $\omega_B = \omega_A, \omega_{rel} = 0$ and so $T = \infty$., particles will maintain their position relative to each other. This is what actually happens in case of geostationary satellite ($\omega_1 = \omega_2$ = constant)

(7) **Angular acceleration (α) :** Angular acceleration of an object in circular motion is defined as the time rate of change of its angular velocity.

(i) If $\Delta\omega$ be the change in angular velocity of the object in time interval Δt, while moving on a circular path, then angular acceleration of the object will be

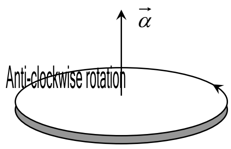


Fig : 3.28

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

(ii) Units : $rad.\ s^{-2}$

(iii) Dimension : $[M^0L^0T^{-2}]$

(iv) Relation between linear acceleration and angular acceleration $\vec{a} = \vec{\alpha} \times \vec{r}$

(v) For uniform circular motion since ω is constant so $\alpha = \frac{d\omega}{dt} = 0$

(vi) For non-uniform circular motion $\alpha \neq 0$

Centripetal Acceleration

(1) Acceleration acting on the object undergoing uniform circular motion is called centripetal acceleration.

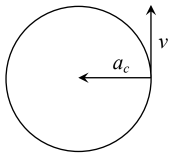


Fig : 3.29

(2) It always acts on the object along the radius towards the centre of the circular path.

(3) Magnitude of centripetal acceleration,

$$a = \frac{v^2}{r} = \omega^2 r = 4\pi^2 n^2 r = \frac{4\pi^2}{T^2} r$$

(4) Direction of centripetal acceleration : It is always the same as that of $\Delta\vec{\upsilon}$. When Δt decreases, $\Delta\theta$ also decreases. Due to which $\Delta\vec{\upsilon}$ becomes more and more perpendicular to $\vec{\upsilon}$. When $\Delta t \to 0$, $\Delta\vec{\upsilon}$ becomes perpendicular to the velocity vector. As the velocity vector of the particle at an instant acts along the tangent to the circular path, therefore $\Delta\vec{\upsilon}$ and hence the centripetal acceleration vector acts along the radius of the circular path at that point and is directed towards the centre of the circular path.

Centripetal force

According to Newton's first law of motion, whenever a body moves in a straight line with uniform velocity, no force is required to maintain this velocity. But when a body moves along a circular path with uniform speed, its direction changes continuously *i.e.* velocity keeps on changing on account of a change in direction. According to Newton's second law of motion, a change in the direction of motion of the body can take place only if some external force acts on the body.

Due to inertia, at every point of the circular path; the body tends to move along the tangent to the circular path at that point (in figure). Since every body has directional inertia, a velocity cannot change by itself and as such we have to apply a force. But this force should be such that it changes the direction of velocity and not its magnitude. This is possible only if the force acts perpendicular to the direction of velocity. Because the velocity is along the tangent, this force must be along the radius (because the radius of a circle at any point is perpendicular to the tangent at that point). Further, as this force is to move the body in a circular path, it must acts towards the centre. This centre-seeking force is called the centripetal force.

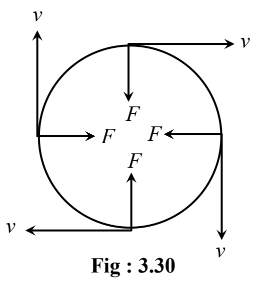


Fig : 3.30

Hence, centripetal force is that force which is required to move a body in a circular path with uniform speed. The force acts on the body along the radius and towards centre.

Formulae for centripetal force :

$$F = \frac{mv^2}{r} = m\omega^2 r = m4\pi^2 n^2 r = \frac{m4\pi^2 r}{T^2}$$

Table 3.1 : Centripetal force in different situation

Situation	Centripetal Force
A particle tied to a string and whirled in a horizontal circle	Tension in the string
Vehicle taking a turn on a level road	Frictional force exerted by the road on the tyres
A vehicle on a speed breaker	Weight of the body or a component of weight
Revolution of earth around the sun	Gravitational force exerted by the sun
Electron revolving around the nucleus in an atom	Coulomb attraction exerted by the protons in the nucleus
A charged particle describing a circular path in a magnetic field	Magnetic force exerted by the agent that sets up the magnetic field

Centrifugal Force

It is an imaginary force due to incorporated effects of inertia. When a body is rotating in a circular path and the centripetal force vanishes, the body would leave the circular path. To an observer *A* who is not sharing the motion along the circular path, the body appears to fly off tangentially at the point of release. To another observer *B*, who is sharing the motion along the circular path (*i.e.*, the observer *B* is also rotating with the body with the same velocity), the body appears to be stationary before it is released. When the body is released, it appears to *B*, as if it has been thrown off along the radius away from the centre by some force. In reality no force is actually seen to act on the body. In absence of any real force the body tends to continue its motion in a straight line due to its inertia. The observer *A* easily relates this events to be due to inertia but since the inertia of both the observer *B* and the body is same, the observer *B* can not relate the above happening to inertia. When the centripetal force ceases to act on the body, the body leaves its circular path and continues to move in its straight-line motion but to observer *B* it appears that a real force has actually acted on the body and is responsible for throwing the body radially out-wards. This imaginary force is given a name to explain the effects of inertia to the observer who is sharing the circular motion of the body. This inertial force is called centrifugal force. Thus centrifugal force is a fictitious force which has significance only in a rotating frame of reference.

Work Done by Centripetal Force

The work done by centripetal force is always zero as it is perpendicular to velocity and hence instantaneous displacement.

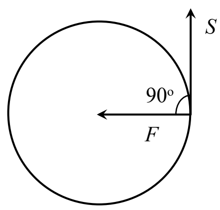


Fig : 3.31

Work done = Increment in kinetic energy of revolving body

Work done = 0

Also $W = \vec{F}.\vec{S} = F \cdot S\cos\theta = F{\cdot}S\cos 90^\circ = 0$

Example : (i) When an electron revolves around the nucleus in hydrogen atom in a particular orbit, it neither absorb nor emit any energy means its energy remains constant.

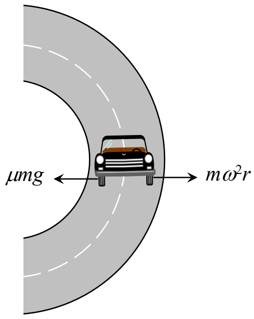
(ii) When a satellite established once in a orbit around the earth and it starts revolving with particular speed, then no fuel is required for its circular motion.

Skidding of Vehicle on A Level Road

When a vehicle takes a turn on a circular path it requires centripetal force.

If friction provides this centripetal force then vehicle can move in circular path safely if

Friction force ≥ Required centripetal force



$$\mu\, mg \geq \frac{mv^2}{r}$$

$$\therefore\ v_{safe} \leq \sqrt{\mu rg}$$

Fig : 3.32

This is the maximum speed by which vehicle can take a turn on a circular path of radius r, where coefficient of friction between the road and tyre is μ.

Skidding of Object on A Rotating Platform

On a rotating platform, to avoid the skidding of an object (mass m) placed at a distance r from axis of rotation, the centripetal force should be provided by force of friction.

Centripetal force ≤ Force of friction

$m\omega^2 r \leq \mu mg$

$$\therefore\ \omega_{\max} = \sqrt{(\mu g / r)},$$

Hence maximum angular velocity of rotation of the platform is $\sqrt{(\mu g / r)}$, so that object will not skid on it.

Bending of A Cyclist

A cyclist provides himself the necessary centripetal force by leaning inward on a horizontal track, while going round a curve. Consider a cyclist of weight mg taking a turn of radius r with velocity v. In order to provide the necessary centripetal force, the cyclist leans through angle θ inwards as shown in figure.

The cyclist is under the action of the following forces :

The weight mg acting vertically downward at the centre of gravity of cycle and the cyclist.

The reaction R of the ground on cyclist. It will act along a line-making angle θ with the vertical.

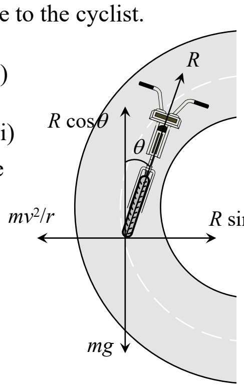
The vertical component $R\cos\theta$ of the normal reaction R will balance the weight of the cyclist, while the horizontal component $R\sin\theta$ will provide the necessary centripetal force to the cyclist.

$$R\sin\theta = \frac{mv^2}{r} \qquad \ldots(i)$$

and $R\cos\theta = mg$ …(ii)

Dividing equation (i) by (ii), we have

$$\frac{R\sin\theta}{R\cos\theta} = \frac{mv^2/r}{mg}$$



or $\tan\theta = \dfrac{v^2}{rg}$ …(iii)

Fig : 3.33

Therefore, the cyclist should bend through an angle $\theta = \tan^{-1}\left(\dfrac{v^2}{rg}\right)$

It follows that the angle through which cyclist should bend will be greater, if

(i) The radius of the curve is small *i.e.* the curve is sharper

(ii) The velocity of the cyclist is large.

Note : ❑ For the same reasons, an ice skater or an aeroplane has to bend inwards, while taking a turn.

Banking of A Road

For getting a centripetal force, cyclist bend towards the centre of circular path but it is not possible in case of four wheelers.

Therefore, outer bed of the road is raised so that a vehicle moving on it gets automatically inclined towards the centre.

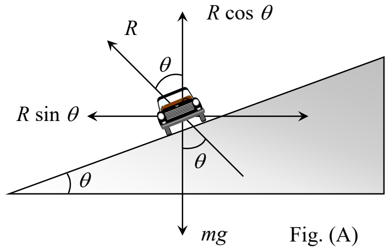


Fig. (A)

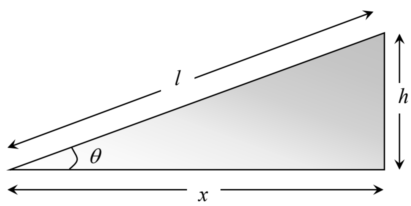


Fig. (B)

Fig : 3.34

In the figure (A) shown reaction R is resolved into two components, the component $R\cos\theta$ balances weight of vehicle

$$\therefore\ R\cos\theta = mg \qquad \ldots(i)$$

and the horizontal component $R\sin\theta$ provides necessary centripetal force as it is directed towards centre of desired circle

Thus $R\sin\theta = \dfrac{mv^2}{r}$ …(ii)

Dividing (ii) by (i), we have

$$\tan\theta = \frac{v^2}{rg} \qquad \ldots(iii)$$

or $\tan\theta = \dfrac{\omega^2 r}{g} = \dfrac{v\omega}{g}$ …(iv) [As $v = r\omega$]

If l = width of the road, h = height of the outer edge from the ground level then from the figure (B)

$\tan\theta = \dfrac{h}{x} = \dfrac{h}{l}$ …(v) [since θ is very small]

From equation (iii), (iv) and (v) $\tan\theta = \frac{v^2}{rg}$

$$= \frac{\omega^2 r}{g} = \frac{v\omega}{g} = \frac{h}{l}$$

Note : ❑ If friction is also present between the tyres and road then $\frac{v^2}{rg} = \frac{\mu + \tan\theta}{1 - \mu\tan\theta}$

❑ Maximum safe speed on a banked frictional road $v = \sqrt{\frac{rg(\mu + \tan\theta)}{1 - \mu\tan\theta}}$

Overturning of Vehicle

When a car moves in a circular path with speed more than a certain maximum speed then it overturns even if friction is sufficient to avoid skidding and its inner wheel leaves the ground first

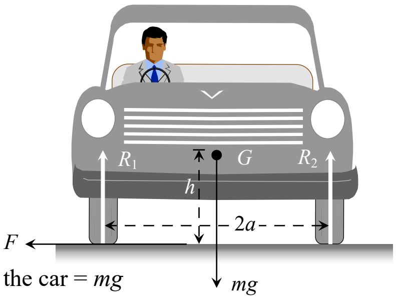


Fig : 3.35

Weight of the car = mg

Speed of the car = v

Radius of the circular path = r

Distance between the centre of wheels of the car = $2a$

Height of the centre of gravity (G) of the car from the road level = h

Reaction on the inner wheel of the car by the ground = R_1

Reaction on the outer wheel of the car by the ground = R_2

When a car move in a circular path, horizontal friction force F provides the required centripetal force

i.e., $F = \frac{mv^2}{R}$ …(i)

For rotational equilibrium, by taking the moment of forces R_1, R_2 and F about G

$$Fh + R_1 a = R_2 a \quad \text{…(ii)}$$

As there is no vertical motion so $R_1 + R_2 = mg$ …(iii)

By solving (i), (ii) and (iii)

$$R_1 = \frac{1}{2}M\left[g - \frac{v^2 h}{ra}\right] \quad \text{…(iv)}$$

and $$R_2 = \frac{1}{2}M\left[g + \frac{v^2 h}{ra}\right] \quad \text{…(v)}$$

It is clear from equation (iv) that if v increases value of R_1 decreases and for $R_1 = 0$

$$\frac{v^2 h}{ra} = g \text{ or } v = \sqrt{\frac{gra}{h}}$$

i.e. the maximum speed of a car without overturning on a flat road is given by $v = \sqrt{\frac{gra}{h}}$

Motion of Charged Particle In Magnetic Field

When a charged particle having mass m , charge q enters perpendicularly in a magnetic field B with velocity v then it describes a circular path.

Because magnetic force (qvB) works in the perpendicular direction of v and it provides required centripetal force

Magnetic force = Centripetal force

$$qvB = \frac{mv^2}{r}$$

∴ radius of the circular path

$$r = \frac{mv}{qB}$$

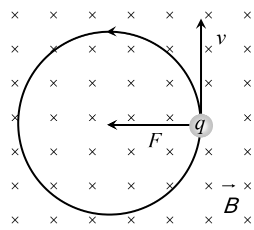


Fig : 3.36

Reaction of Road On Car

(1) When car moves on a concave bridge then

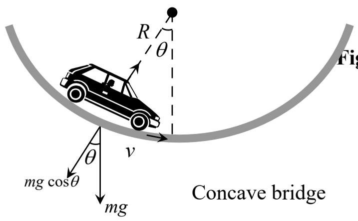


Fig : 3.37

Centripetal force = $R - mg\cos\theta = \frac{mv^2}{r}$

and reaction $R = mg\cos\theta + \frac{mv^2}{r}$

(2) When car moves on a convex bridge

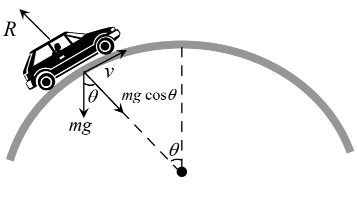


Fig : 3.38

Centripetal force = $mg\cos\theta - R = \frac{mv^2}{r}$

and reaction $R = mg\cos\theta - \frac{mv^2}{r}$

Non-Uniform Circular Motion

If the speed of the particle in a horizontal circular motion changes with respect to time, then its motion is said to be non-uniform circular motion.

Consider a particle describing a circular path of radius r with centre at O. Let at an instant the particle be at P and $\vec{v}$ be its linear velocity and $\vec{\omega}$ be its angular velocity.

Then, $\vec{\upsilon} = \vec{\omega} \times \vec{r}$ …(i)

Differentiating both sides of *w.r.t.* time *t* we have

$$\frac{d\vec{\upsilon}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} \quad \text{…(ii)}$$

Here, $\frac{d\vec{v}}{dt} = \vec{a}$, (Resultant acceleration)

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\upsilon}$$

$\frac{d\vec{\omega}}{dt} = \vec{\alpha}$ (Angular acceleration)

$$\vec{a} = \vec{a}_t + \vec{a}_c \quad \text{…(iii)}$$

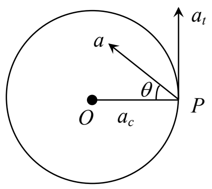


Fig : 3.39

$\frac{d\vec{r}}{dt} = \vec{\upsilon}$ (Linear velocity)

Thus the resultant acceleration of the particle at *P* has two component accelerations

(1) **Tangential acceleration :** $\vec{a}_t = \vec{\alpha} \times \vec{r}$

It acts along the tangent to the circular path at *P* in the plane of circular path.

According to right hand rule since $\vec{\alpha}$ and $\vec{r}$ are perpendicular to each other, therefore, the magnitude of tangential acceleration is given by

$$|\vec{a}_t| = |\vec{\alpha} \times \vec{r}| = \alpha r \sin 90^o = \alpha r.$$

(2) **Centripetal (Radial) acceleration** : $\vec{a}_c = \vec{\omega} \times \vec{v}$

It is also called centripetal acceleration of the particle at *P*.

It acts along the radius of the particle at *P*.

According to right hand rule since $\vec{\omega}$ and $\vec{\upsilon}$ are perpendicular to each other, therefore, the magnitude of centripetal acceleration is given by

$$|\vec{a}_c| = |\vec{\omega} \times \vec{\upsilon}| = \omega \upsilon \sin 90^o = \omega \upsilon = \omega(\omega r) = \omega^2 r = \upsilon^2 / r$$

Table 3.2 : Tangential and centripetal acceleration

Centripetal acceleration	Tangential acceleration	Net acceleration	Type of motion
$a_c = 0$	$a_t = 0$	$a = 0$	Uniform translatory motion
$a_c = 0$	$a_t \neq 0$	$a = a_t$	Accelerated translatory motion
$a_c \neq 0$	$a_t = 0$	$a = a_c$	Uniform circular motion
$a_c \neq 0$	$a_t \neq 0$	$a = \sqrt{a_c^2 + a_t^2}$	Non-uniform circular motion

Note : ❑ Here a_t governs the magnitude of $\vec{v}$ while $\vec{a}_c$ its direction of motion.

(3) **Force :** In non-uniform circular motion the particle simultaneously possesses two forces

Centripetal force : $F_c = ma_c = \frac{mv^2}{r} = mr\omega^2$

Tangential force : $F_t = ma_t$

Net force : $F_{net} = ma = m\sqrt{a_c^2 + a_t^2}$

Note : ❑ In non-uniform circular motion work done by centripetal force will be zero since $\vec{F}_c \perp \vec{v}$

❑ In non uniform circular motion work done by tangential force will not be zero since $F_t \neq 0$

❑ Rate of work done by net force in non-uniform circular motion = rate of work done by tangential force

i.e. $P = \frac{dW}{dt} = \vec{F}_t.\vec{v}$

Equations of Circular Motion

For accelerated motion	For retarded motion
$\omega_2 = \omega_1 + \alpha t$	$\omega_2 = \omega_1 - \alpha t$
$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$	$\theta = \omega_1 t - \frac{1}{2}\alpha t^2$
$\omega_2^2 = \omega_1^2 + 2\alpha\theta$	$\omega_2^2 = \omega_1^2 - 2\alpha\theta$
$\theta_n = \omega_1 + \frac{\alpha}{2}(2n-1)$	$\theta_n = \omega_1 - \frac{\alpha}{2}(2n-1)$

Where

ω_1 = Initial angular velocity of particle

ω_2 = Final angular velocity of particle

α = Angular acceleration of particle

θ = Angle covered by the particle in time *t*

θ_n = Angle covered by the particle in n^{th} second

Motion in vertical circle

This is an example of non-uniform circular motion. In this motion body is under the influence of gravity of earth. When body moves from lowest point to highest point. Its speed decrease and becomes minimum at highest point. Total mechanical energy of the body remains conserved and *KE* converts into *PE* and vice versa.

(1) **Velocity at any point on vertical loop :** If *u* is the initial velocity imparted to body at lowest point then velocity of body at height *h* is given by

$$v = \sqrt{u^2 - 2gh} = \sqrt{u^2 - 2gl(1 - \cos\theta)}$$

[As $h = l - l\cos\theta = l(1 - \cos\theta)$]

where *l* is the length of the string

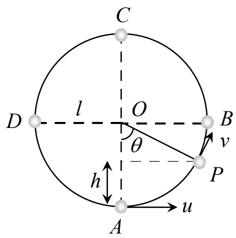


Fig : 3.40

(2) **Tension at any point on vertical loop** : Tension at general point *P*, According to Newton's second law of motion.

Net force towards centre = centripetal force

$$T - mg\cos\theta = \frac{mv^2}{l}$$

or $T = mg\cos\theta + \frac{mv^2}{l}$

$$T = \frac{m}{l}[u^2 - gl(2 - 3\cos\theta)]$$

[As $v = \sqrt{u^2 - 2gl(1-\cos\theta)}$]

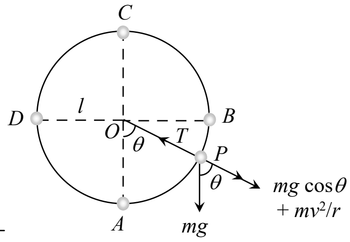


Fig : 3.41

Table 3.3 : Velocity and tension in a vertical loop

Position	Angle	Velocity	Tension
A	0°	*u*	$\frac{mu^2}{l} + mg$
B	90°	$\sqrt{u^2 - 2gl}$	$\frac{mu^2}{l} - 2mg$
C	180°	$\sqrt{u^2 - 4gl}$	$\frac{mu^2}{l} - 5mg$
D	270°	$\sqrt{u^2 - 2gl}$	$\frac{mu^2}{l} - 2mg$

It is clear from the table that : $T_A > T_B > T_C$ and $T_B = T_D$

$T_A - T_B = 3mg,$

$T_A - T_C = 6mg$

and $T_B - T_C = 3mg$

Table 3.4 : Various conditions for vertical motion

Velocity at lowest point	Condition
$u_A > \sqrt{5gl}$	Tension in the string will not be zero at any of the point and body will continue the circular motion.
$u_A = \sqrt{5gl},$	Tension at highest point C will be zero and body will just complete the circle.
$\sqrt{2gl} < u_A < \sqrt{5gl},$	Particle will not follow circular motion. Tension in string become zero somewhere between points *B* and *C* whereas velocity remain positive. Particle leaves circular path and follow parabolic trajectory.
$u_A = \sqrt{2gl}$	Both velocity and tension in the string becomes zero at *B* and particle will oscillate along semi-circular path.
$u_A < \sqrt{2gl}$	velocity of particle becomes zero between *A* and *B* but tension will not be zero and the particle will oscillate about the point *A*.

Note : ❑ *K.E.* of a body moving in horizontal circle is same throughout the path but the *K.E.* of the body moving in vertical circle is different at different places.

❑ If body of mass *m* is tied to a string of length *l* and is projected with a horizontal velocity *u* then :

Height at which the velocity vanishes is $h = \frac{u^2}{2g}$

Height at which the tension vanishes is $h = \frac{u^2 + gl}{3g}$

(3) **Critical condition for vertical looping :** If the tension at *C* is zero, then body will just complete revolution in the vertical circle. This state of body is known as critical state. The speed of body in critical state is called as critical speed.

From the above table 3.3 $T_C = \frac{mu^2}{l} - 5mg = 0$

$\Rightarrow u = \sqrt{5gl}$

It means to complete the vertical circle the body must be projected with minimum velocity of $\sqrt{5gl}$ at the lowest point.

Table 3.5 : Different variables in vertical loop

Quantity	Point *A*	Point *B*	Point *C*	Point *D*	Point *P*
Linear velocity (v)	$\sqrt{5gl}$	$\sqrt{3gl}$	$\sqrt{gl}$	$\sqrt{3gl}$	$\sqrt{gl(3+2\cos\theta)}$
Angular velocity (ω)	$\sqrt{\frac{5g}{l}}$	$\sqrt{\frac{3g}{l}}$	$\sqrt{\frac{g}{l}}$	$\sqrt{\frac{3g}{l}}$	$\sqrt{\frac{g}{l}(3+2\cos\theta)}$
Tension in String (T)	$6\ mg$	$3\ mg$	0	$3\ mg$	$3mg(1+\cos\theta)$
Kinetic Energy (KE)	$\frac{5}{2}mgl$	$\frac{3}{2}mgl$	$\frac{1}{2}mgl$	$\frac{3}{2}mgl$	$\frac{mu^2}{l}-5mg=0$
Potential Energy (PE)	0	mgl	$2\ mgl$	mgl	$mgl(1-\cos\theta)$
Total Energy (TE)	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$

(4) **Motion of a block on frictionless hemisphere** : A small block of mass *m* slides down from the top of a frictionless hemisphere of radius *r*. The component of the force of gravity ($mg\cos\theta$) provides required centripetal force but at point *B* it's circular motion ceases and the block lose contact with the surface of the sphere.

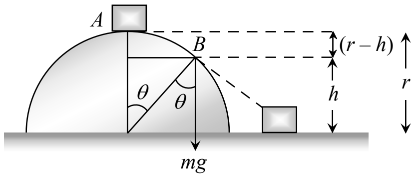


Fig : 3.42

For point *B*, by equating the forces,

$$mg\cos\theta = \frac{mv^2}{r} \quad \text{...(i)}$$

For point *A* and *B*, by law of conservation of energy

Total energy at point *A* = Total energy at point *B*

$K.E._{(A)} + P.E._{(A)} = K.E._{(B)} + P.E._{(B)}$

$$0 + mgr = \frac{1}{2}mv^2 + mgh \Rightarrow v = \sqrt{2g(r-h)} \quad \text{...(ii)}$$

and from the given figure $h = r\cos\theta$ ...(iii)

By substituting the value of *v* and *h* from eq^n (ii) and (iii) in eq^n (i)

$$mg\left(\frac{h}{r}\right) = \frac{m}{r}\left(\sqrt{2g(r-h)}\right)^2 \Rightarrow h = 2(r-h) \Rightarrow h = \frac{2}{3}r$$

i.e. the block lose contact at the height of $\frac{2}{3}r$ from the ground.

and angle from the vertical can be given by $\cos\theta = \frac{h}{r} = \frac{2}{3}$

$\therefore\ \theta = \cos^{-1}\frac{2}{3}$.

Conical Pendulum

This is the example of uniform circular motion in horizontal plane.

A bob of mass *m* attached to a light and in-extensible string rotates in a horizontal circle of radius *r* with constant angular speed ω about the vertical. The string makes angle θ with vertical and appears tracing the surface of a cone. So this arrangement is called conical pendulum.

The force acting on the bob are tension and weight of the bob.

From the figure $T\sin\theta = \frac{mv^2}{r}$ ....(i)

and $T\cos\theta = mg$ ....(ii)

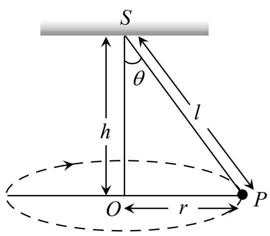


Fig : 3.43

(1) Tension in the string : $T = mg\sqrt{1+\left(\frac{v^2}{rg}\right)^2}$

$$T = \frac{mg}{\cos\theta} = \frac{mgl}{\sqrt{l^2 - r^2}} \qquad \left[\text{As } \cos\theta = \frac{h}{l} = \frac{\sqrt{l^2-r^2}}{l}\right]$$

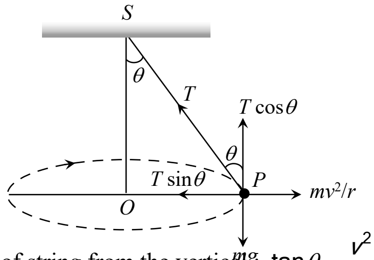


Fig : 3.44

(2) Angle of string from the vertical : $\tan\theta = \frac{v^2}{rg}$

(3) Linear velocity of the bob : $v = \sqrt{gr\tan\theta}$

(4) Angular velocity of the bob :

$$\omega = \sqrt{\frac{g}{r}\tan\theta} = \sqrt{\frac{g}{h}} = \sqrt{\frac{g}{l\cos\theta}}$$

(5) Time period of revolution :

$$T_P = 2\pi\sqrt{\frac{l\cos\theta}{g}} = 2\pi\sqrt{\frac{h}{g}}$$

$$= 2\pi\sqrt{\frac{\sqrt{l^2-r^2}}{g}} = 2\pi\sqrt{\frac{r}{g\tan\theta}}$$

Tips & Tricks

✍ Consider a projectile of mass *m* thrown with velocity *u* making angle θ with the horizontal. It is projected from the point *O* and returns to the ground at *G*. Also *M* is the highest point attained by it. (See figure).

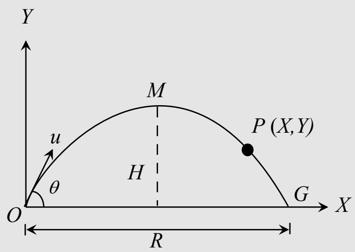


Fig : 3.45

(i) In going from *O* to *M*, following changes take place –

(a) Change in velocity $= u\sin\theta$

(b) Change in speed $= u(1-\cos\theta) = 2u\cos^2(\theta/2)$

(c) Change in momentum $= mu\sin\theta$

(d) Change (loss) in kinetic energy $= 1/2\, mu^2\sin^2\theta$

(e) Change (gain) in potential energy $= 1/2\, mu^2\sin^2\theta$

(f) Change in the direction of motion $= \angle\theta$

(ii) On return to the ground, that is in going from O to G, the following changes take place

(a) Change in speed = zero

(b) Change in velocity = $2u\sin\theta$

(c) Change in momentum = $2mu\sin\theta$

(d) Change in kinetic energy = zero

(e) Change in potential energy = zero

(f) Change in the direction of motion = $\angle 2\theta$

✍ (i) At highest point, the horizontal component of velocity is $v_x = u\cos\theta$ and vertical component of velocity v_y is zero.

(ii) At highest point, linear momentum of a particle

$m v_x = mu\cos\theta$.

(iii) Kinetic energy of the particle at the highest point = $\frac{1}{2}mv_x^2$

$= \frac{1}{2}mu^2\cos^2\theta$.

✍ At highest point, acceleration due to gravity acting vertically downward makes an angle of 90° with the horizontal component of the velocity of the projectile.

✍ At the highest point, momentum of the projectile thrown at an angle θ with horizontal is $p\cos\theta$ and K.E. = $(\text{K.E.})_i\cos^2\theta$.

✍ In projectile motion, horizontal component $u\cos\theta$ of velocity u remains constant throughout, whereas vertical component $u\sin\theta$ changes and becomes zero at the highest point.

✍ The trajectory of a projectile is parabolic.

✍ For a projectile, time of flight and maximum height depend on the vertical component of the velocity of projection.

✍ The range of the projectile is maximum for the angle of projection $\theta = 45°$.

✍ The maximum range of the projectile is :

$$R_{max} = \frac{u^2}{g}$$

✍ When the range is maximum, the height attained by the projectile is :

$$H = \frac{u^2}{4g} = \frac{R_{max}}{4}$$

✍ When the range of the projectile is maximum, the time of flight is :

$$T = 2t = \frac{\sqrt{2}u}{g}$$

✍ The height attained by a projectile is maximum, when $\theta = 90°$.

$$H_{max} = \frac{u^2}{2g}$$

It is twice that of height attained, when the range is maximum.

✍ The time of flight of the projectile is also largest for $\theta = 90°$.

$$T_{max} = \frac{2u}{g}$$

✍ The trajectory of the projectile is a symmetric parabola only when g is constant through out the motion and θ is not equal to 0°, 90° or 180°.

✍ If velocity of projection is made n times, the maximum height attained and the range become n^2 times and the time of flight becomes n times the initial value.

✍ If the force acting on a particle is always perpendicular to the velocity of the particle, then the path of the particle is a circle. The centripetal force is always perpendicular to the velocity of the particle.

✍ If circular motion of the object is uniform, the object will possess only centripetal acceleration.

✍ If circular motion of the object is non-uniform, the object will possess both centripetal and transverse acceleration.

✍ When the particle moves along the circular path with constant speed, the angular velocity is also constant. But linear velocity, momentum as well as centripetal acceleration change in direction, although their magnitude remains unchanged.

✍ For circular motion of rigid bodies with uniform speed, the angular speed is same for all particles, but linear speed varies directly as the radius of the circular path described by the particle $(v \propto r)$.

✍ When a body rotates, all its particles describe circular paths about a line, called axis of rotation.

✍ The centre of the circle describe by the different particles of the rotating body lie on the axis of rotation.

✍ Centripetal force $F_c = ma_c$, $m\omega^2 r$ where m = mass of the body.

✍ Centripetal force is always directed towards the centre of the circular path.

✍ When a body rotates with uniform velocity, its different particles have centripetal acceleration directly proportional to the radius $(a_c \propto r)$.

✍ There can be no circular motion without centripetal force.

✍ Centripetal force can be mechanical, electrical or magnetic force.

✍ Planets go round the earth in circular orbits due to the centripetal force provided by gravitational force of the sun.

✍ Gravitational pull of earth provides centripetal force for the orbital motion of the moon and artificial satellites.

✍ Centripetal force cannot change the kinetic energy of the body.

✍ In uniform circular motion the magnitude of the centripetal acceleration remains constant whereas its direction changes continuously but always directed towards the centre.

✍ A pseudo force, that is equal and opposite to the centripetal force is called centrifugal force.

✍ The $\vec{\theta}$, $\vec{\omega}$ and $\vec{\alpha}$ are directed along the axis of the circular path. Their sense of direction is given by the right hand fist rule as follows : 'If we catch axis of rotation in right hand fist such that the fingers point in the direction of rotation, then the outstretched thumb gives the direction of $\vec{\theta}$, $\vec{\omega}$ and $\vec{\alpha}$

✍ $\vec{\theta}$, $\vec{\omega}$ and $\vec{\alpha}$ are called pseudo vectors or axial vectors.

✍ For circular motion we have –

(i) $\vec{r} \perp \vec{v}$ (ii) $\vec{r}$ antiparallel to $\vec{a}_c$

(iii) $\vec{a}_c \perp \vec{v}$ (iv) $\vec{a}_c \perp \vec{a}_t$

(v) $\vec{\theta}$, $\vec{\omega}$, $\vec{\alpha}$ are perpendicular to $\vec{r}, \vec{a}_c, \vec{a}_t, \vec{v}$

(vi) $\vec{r}, \vec{a}_c, \vec{a}_t$ and $\vec{v}$ lie in the same plane