Sample problem based on moment of inertia

***Problem* 14.** Five particles of mass = 2 *kg* are attached to the rim of a circular disc of radius 0.1 *m* and negligible mass. Moment of inertia of the system about the axis passing through the centre of the disc and perpendicular to its plane is

(a) 1 $kg\ m^2$ (b) 0.1 $kg\ m^2$ (c) 2 $kg\ m^2$ (d) 0.2 $kg\ m^2$

Solution: (b) We will not consider the moment of inertia of disc because it doesn't have any mass so moment of inertia of five particle system $I = 5\,mr^2 = 5 \times 2 \times (0.1)^2 = 0.1\,kg\text{-}m^2$.

***Problem* 15.** A circular disc X of radius R is made from an iron plate of thickness t, and another disc Y of radius $4R$ is made from an iron plate of thickness $\frac{t}{4}$. Then the relation between the moment of inertia I_X and I_Y is **[AIEEE 2003]**

(a) $I_Y = 64 I_X$ (b) $I_Y = 32 I_X$ (c) $I_Y = 16 I_X$ (d) $I_Y = I_X$

Solution: (a) Moment of Inertia of disc $I = \frac{1}{2}MR^2 = \frac{1}{2}(\pi R^2 t\rho)R^2 = \frac{1}{2}\pi t\rho R^4$

[As $M = V \times \rho = \pi R^2 t\rho$ where $t =$ thickness, $\rho =$ density]

$\therefore \quad \frac{I_y}{I_x} = \frac{t_y}{t_x}\left(\frac{R_y}{R_x}\right)^4$ [If ρ = constant]

$\Rightarrow \quad \frac{I_y}{I_x} = \frac{1}{4}(4)^4 = 64$ [Given $R_y = 4R_x$, $t_y = \frac{t_x}{4}$]

$\Rightarrow \quad I_y = 64\,I_x$

***Problem* 16.** Moment of inertia of a uniform circular disc about a diameter is *I*. Its moment of inertia about an axis perpendicular to its plane and passing through a point on its rim will be

(a) 5 *I* (b) 6 *I* (c) 3 *I* (d) 4 *I*

Solution: (b) Moment of inertia of disc about a diameter $= \frac{1}{4}MR^2 = I$ (given) $\therefore\ MR^2 = 4I$

Now moment of inertia of disc about an axis perpendicular to its plane and passing through a point on its rim

$$= \frac{3}{2}MR^2 = \frac{3}{2}(4I) = 6I.$$

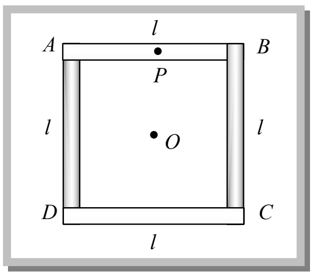
***Problem* 17.** Four thin rods of same mass M and same length l, form a square as shown in figure. Moment of inertia of this system about an axis through centre O and perpendicular to its plane is

(a) $\frac{4}{3}Ml^2$

(b) $\frac{Ml^2}{3}$

(c) $\frac{Ml^2}{6}$

(d) $\frac{2}{3}Ml^2$

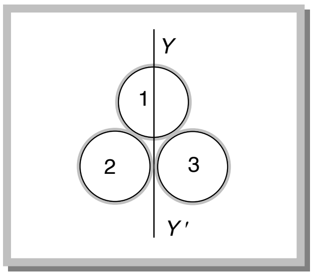


Solution: (a) Moment of inertia of rod AB about point $P = \frac{1}{12}Ml^2$

M.I. of rod AB about point $O = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2$ [by the theorem of parallel axis]

and the system consists of 4 rods of similar type so by the symmetry $I_{System} = \frac{4}{3}Ml^2$.

***Problem* 18.** Three rings each of mass M and radius R are arranged as shown in the figure. The moment of inertia of the system about YY' will be



(a) $3\,MR^2$

(b) $\frac{3}{2}MR^2$

(c) $5\,MR^2$

(d) $\frac{7}{2}MR^2$

Solution: (d) M.I of system about YY' $I = I_1 + I_2 + I_3$

where I_1 = moment of inertia of ring about diameter, $I_2 = I_3$ = M.I. of inertia of ring about a tangent in a plane

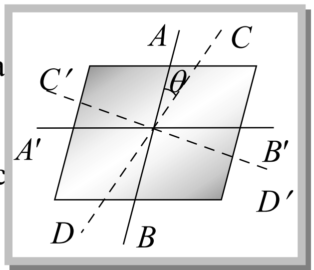
$\therefore \quad I = \frac{1}{2}mR^2 + \frac{3}{2}mR^2 + \frac{3}{2}mR^2 = \frac{7}{2}mR^2$

***Problem* 19.** Let I be the moment of inertia of an uniform square plate about an axis AB that passes through its centre and is parallel to two of its sides. CD is a line in the plane of the plate that passes through the centre of the plate and makes an angle θ with AB. The moment of inertia of the plate about the axis CD is then equal to

[IIT-JEE 1998]

(a) I (b) $I\sin^2\theta$ (c) $I\cos^2\theta$ (d) $I\cos^2\frac{\theta}{2}$

Solution: (a) Let I_Z is the moment of inertia of square plate about the axis which is passing through the centre and perpendicular to the plane.



$I_Z = I_{AB} + I_{A'B'} = I_{CD} + I_{C'D'}$ [By the theorem of perpendicular a

$I_Z = 2I_{AB} = 2I_{A'B'} = 2I_{CD} = 2I_{C'D'}$

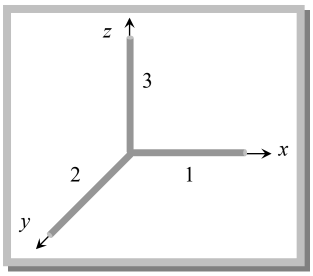
[As AB, $A'B'$ and CD, $C'D'$ are symmetric

Hence $I_{CD} = I_{AB} = I$

***Problem* 20.** Three rods each of length L and mass M are placed along X, Y and Z-axes in such a way that one end of each of the rod is at the origin. The moment of inertia of this system about Z axis is

(a) $\frac{2ML^2}{3}$ (b) $\frac{4ML^2}{3}$ (c) $\frac{5ML^2}{3}$ (d) $\frac{ML^2}{3}$

Solution: (a) Moment of inertia of the system about z-axis can be find out by calculating the moment of inertia of individual rod about z-axis



$I_1 = I_2 = \frac{ML^2}{3}$ because z-axis is the edge of rod 1 and 2

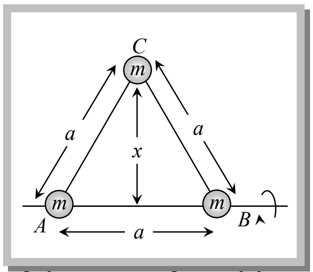
and $I_3 = 0$ because rod in lying on z-axis

$\therefore \; I_{system} = I_1 + I_2 + I_3 = \frac{ML^2}{3} + \frac{ML^2}{3} + 0 = \frac{2ML^2}{3}$.

<u>Problem</u> 21. Three point masses each of mass m are placed at the corners of an equilateral triangle of side a. Then the moment of inertia of this system about an axis passing along one side of the triangle is **[AIIMS 19**

(a) ma^2 (b) $3ma^2$ (c) $\frac{3}{4}ma^2$ (d) $\frac{2}{3}ma^2$

Solution: (c) The moment of inertia of system about AB side of triangle

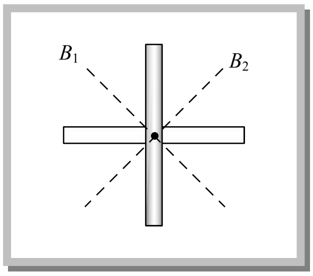


$I = I_A + I_B + I_C$

$= 0 + 0 + mx^2$

$= m\left(\frac{a\sqrt{3}}{2}\right)^2 = \frac{3}{4}ma^2$

<u>Problem</u> 22. Two identical rods each of mass M. and length l are joined in crossed position as shown in figure. The moment of inertia of this system about a bisector would be



(a) $\frac{Ml^2}{6}$

(b) $\frac{Ml^2}{12}$

(c) $\frac{Ml^2}{3}$

(d) $\frac{Ml^2}{4}$

Solution: (b) Moment of inertia of system about an axes which is perpendicular to plane of rods and passing through the common centre of rods $I_z = \frac{Ml^2}{12} + \frac{Ml^2}{12} = \frac{Ml^2}{6}$

Again from perpendicular axes theorem $I_z = I_{B_1} + I_{B_2} = 2I_{B_1} = 2I_{B_2} = \frac{Ml^2}{6}$ [As $I_{B_1} = I_{B_2}$]

$\therefore \quad I_{B_1} = I_{B_2} = \frac{Ml^2}{12}$.

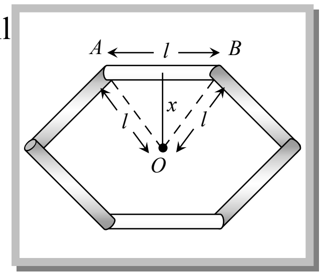
<u>Problem</u> 23. The moment of inertia of a rod of length l about an axis passing through its centre of mass and perpendicular to rod is I. The moment of inertia of hexagonal shape formed by six such rods, about an axis passing through its centre of mass and perpendicular to its plane will be

(a) $16I$ (b) $40\,I$ (c) $60\,I$ (d) $80\,I$

Solution: (c) Moment of inertia of rod AB about its centre and perpendicular to the length $= \frac{ml^2}{12} = I \quad \therefore$

$ml^2 = 12I$

Now moment of inertia of the rod about the axis which is passing through O and perpendicular to the plane of hexagon $I_{rod} = \frac{ml^2}{12} + mx^2$ [From the theorem of parall



$= \frac{ml^2}{12} + m\left(\frac{\sqrt{3}}{2}l\right)^2 = \frac{5ml^2}{6}$

Now the moment of inertia of system $I_{system} = 6\times I_{rod} = 6\times\frac{5ml^2}{6} = 5ml^2$

$I_{system} = 5\,(12\,I) = 60\,I$ [As $ml^2 = 12I$]

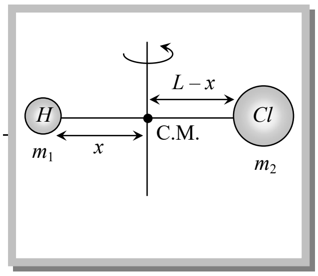
Problem 24. The moment of inertia of HCl molecule about an axis passing through its centre of mass and perpendicular to the line joining the H^+ and Cl^- ions will be, if the interatomic distance is 1 Å

(a) $0.61\times10^{-47}\ kg.m^2$ (b) $1.61\times10^{-47}\ kg.m^2$ (c) $0.061\times10^{-47}\ kg.m^2$ (d) 0

Solution: (b) If r_1 and r_2 are the respective distances of particles m_1 and m_2 from the centre of mass then

$m_1r_1 = m_2r_2 \Rightarrow 1\times x = 35.5\times(L-x) \Rightarrow x = 35.5(1-x)$

$\Rightarrow x = 0.973\text{Å}$ and $L-x = 0.027\text{Å}$



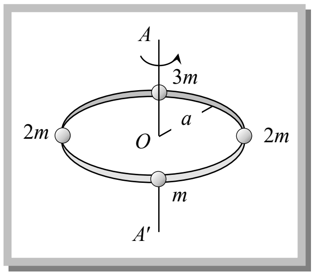
Moment of inertia of the system about centre of mass $I = m_1x^2 + m_2(L-x)^2$

$I = 1amu\times(0.973\text{Å})^2 + 35.5\ amu\times(0.027\text{Å})^2$

Substituting 1 *a.m.u.* = 1.67 × 10^{-27} *kg* and 1 Å = 10^{-10} *m*

$I = 1.62\times10^{-47}\ kg m^2$

Problem 25. Four masses are joined to a light circular frame as shown in the figure. The radius of gyration of this system about an axis passing through the centre of the circular frame and perpendicular to its plane would be



(a) $a/\sqrt{2}$

(b) $a/2$

(c) a

(d) $2a$

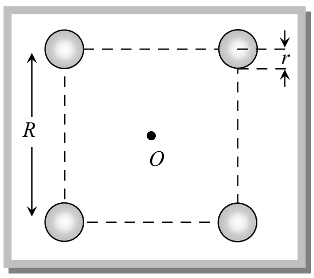
Solution: (c) Since the circular frame is massless so we will consider moment of inertia of four masses only.

$I = ma^2 + 2ma^2 + 3ma^2 + 2ma^2 = 8ma^2$ .....(i)

Now from the definition of radius of gyration $I = 8mk^2$ .....(ii)

comparing (i) and (ii) radius of gyration $k = a$.

Problem 26. Four spheres, each of mass M and radius r are situated at the four corners of square of side R. The moment of inertia of the system about an axis perpendicular to the plane of square and passing through its centre will be



(a) $\frac{5}{2}M(4r^2+5R^2)$

(b) $\frac{2}{5}M(4r^2+5R^2)$

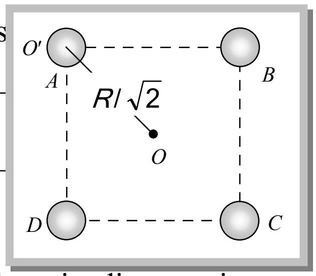
(c) $\frac{2}{5}M(4r^2+5r^2)$

(d) $\frac{5}{2}M(4r^2+5r^2)$

Solution: (b) *M. I.* of sphere *A* about its diameter $I_{O'} = \frac{2}{5}Mr^2$

Now *M.I.* of sphere *A* about an axis perpendicular to the plane of square and passing through its centre will be

$I_O = I_{O'} + M\left(\frac{R}{\sqrt{2}}\right)^2 = \frac{2}{5}Mr^2 + \frac{MR^2}{2}$ [by the theorem of parallel axis]



Moment of inertia of system (i.e. four sphere)= $4I_O = 4\left[\frac{2}{5}Mr^2 + \frac{MR^2}{2}\right]$

$= \frac{2}{5}M\left[4r^2 + 5R^2\right]$

***Problem* 27.** The moment of inertia of a solid sphere of density ρ and radius *R* about its diameter is

(a) $\frac{105}{176}R^5\rho$ (b) $\frac{105}{176}R^2\rho$ (c) $\frac{176}{105}R^5\rho$ (d) $\frac{176}{105}R^2\rho$

Solution: (c) Moment of inertia of sphere about it diameter $I = \frac{2}{5}MR^2 = \frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2$ [As $M = V\rho = \frac{4}{3}\pi R^3\rho$]

$I = \frac{8\pi}{15}R^5\rho = \frac{8\times 22}{15\times 7}R^5\rho = \frac{176}{105}R^5\rho$

***Problem* 28.** Two circular discs A and B are of equal masses and thickness but made of metals with densities d_A and d_B $(d_A > d_B)$. If their moments of inertia about an axis passing through centres and normal to the circular faces be I_A and I_B, then

(a) $I_A = I_B$ (b) $I_A > I_B$ (c) $I_A < I_B$ (d) $I_A >=< I_B$

Solution : (c) Moment of inertia of circular disc about an axis passing through centre and normal to the circular face

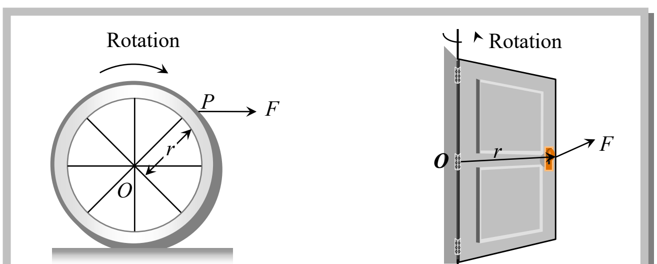
$I = \frac{1}{2}MR^2 = \frac{1}{2}M\left(\frac{M}{\pi t\rho}\right)$ [As $M = V\rho = \pi R^2 t\rho \;\therefore\; R^2 = \frac{M}{\pi t\rho}$]

$\Rightarrow$ $I = \frac{M^2}{2\pi t\rho}$ or $I \propto \frac{1}{\rho}$ If mass and thickness are constant.

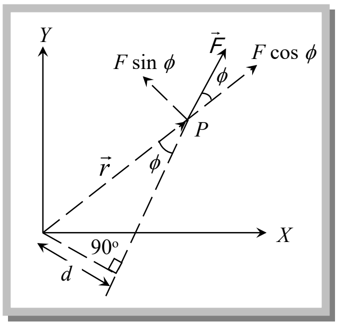
So, in the problem $\frac{I_A}{I_B} = \frac{d_B}{d_A}$ $\therefore I_A < I_B$ [As $d_A > d_B$]

7.14 Torque

If a pivoted, hinged or suspended body tends to rotate under the action of a force, it is said to be acted upon by a torque. or The turning effect of a force about the axis of rotation is called moment of force or torque due to the force.



(A) (B)



If the particle rotating in xy plane about the origin under the effect of force $\vec{F}$ and at any instant the position vector of the particle is $\vec{r}$ then,

$$\text{Torque } \vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF\sin\phi$$

[where ϕ is the angle between the direction of $\vec{r}$ and $\vec{F}$]

(1) Torque is an axial vector. *i.e.,* its direction is always perpendicular to the plane containing vector $\vec{r}$ and $\vec{F}$ in accordance with right hand screw rule. For a given figure the sense of rotation is anti-clockwise so the direction of torque is perpendicular to the plane, outward through the axis of rotation.

(2) Rectangular components of force

$$\vec{F}_r = F\cos\phi = \text{radial component of force}, \quad \vec{F}_\phi = F\sin\phi = \text{transverse component of force}$$

As $\tau = rF\sin\phi$

or $\tau = rF_\phi$ = (position vector) × (transverse component of force)

Thus the magnitude of torque is given by the product of transverse component of force and its perpendicular distance from the axis of rotation *i.e.*, Torque is due to transverse component of force only.

(3) As $\tau = rF\sin\phi$

or $\tau = F(r\sin\phi) = Fd$ [As $d = r\sin\phi$ from the figure]

i.e. Torque = Force × Perpendicular distance of line of action of force from the axis of rotation.

Torque is also called as moment of force and *d* is called moment or lever arm.

(4) Maximum and minimum torque : As $\vec{\tau} = \vec{r} \times \vec{F}$ or $\tau = rF\sin\phi$

$\tau_{maximum} = rF$	When $\lvert\sin\phi\rvert = \max = 1$ *i.e.,* $\phi = 90°$	$\vec{F}$ is perpendicular to $\vec{r}$
$\tau_{minimum} = 0$	When $\lvert\sin\phi\rvert = \min = 0$ *i.e.* $\phi = 0°$ or $180°$	$\vec{F}$ is collinear to $\vec{r}$

(5) For a given force and angle, magnitude of torque depends on *r*. The more is the value of *r*, the more will be the torque and easier to rotate the body.

Example : (i) Handles are provided near the free edge of the Planck of the door.

(ii) The handle of screw driver is taken thick.

(iii) In villages handle of flourmill is placed near the circumference.

(iv) The handle of hand-pump is kept long.

(v) The arm of wrench used for opening the tap, is kept long.

(6) Unit : *Newton-metre* (M.K.S.) and *Dyne-cm* (C.G.S.)

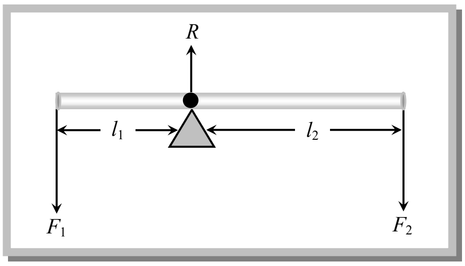
(7) Dimension : $[ML^2T^{-2}]$.

(8) If a body is acted upon by more than one force, the total torque is the vector sum of each torque.

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \ldots\ldots$$

(9) A body is said to be in rotational equilibrium if resultant torque acting on it is zero *i.e.* $\Sigma \vec{\tau} = 0$.

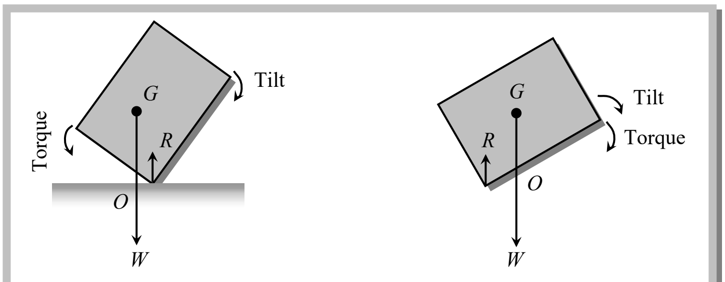
(10) In case of beam balance or see-saw the system will be in rotational equilibrium if,



$\vec{\tau}_1 + \vec{\tau}_2 = 0$ or $F_1 l_1 - F_2 l_2 = 0 \;\therefore\; F_1 l_1 = F_2 l_2$

However if, $\vec{\tau}_1 > \vec{\tau}_2$, L.H.S. will move downwards and if $\vec{\tau}_1 < \vec{\tau}_2$. R.H.S. will move downward. and the system will not be in rotational equilibrium.

(11) On tilting, a body will restore its initial position due to torque of weight about the point *O* till the line of action of weight passes through its base on tilting, a body will topple due to torque of weight about *O*, if the line of action of weight does not pass through the base.



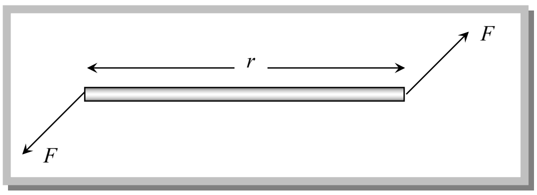
(12) Torque is the cause of rotatory motion and in rotational motion it plays same role as force plays in translatory motion *i.e.,* torque is rotational analogue of force. This all is evident from the following correspondences between rotatory and translatory motion.

Rotatory Motion	Translatory Motion
$\vec{\tau} = I\vec{\alpha}$	$\vec{F} = m\vec{a}$
$W = \int \vec{\tau} \cdot d\vec{\theta}$	$W = \int \vec{F} \cdot d\vec{s}$
$P = \vec{\tau} \cdot \vec{\omega}$	$P = \vec{F} \cdot \vec{v}$
$\vec{\tau} = \frac{d\vec{L}}{dt}$	$\vec{F} = \frac{d\vec{P}}{dt}$

7.15 Couple

A special combination of forces even when the entire body is free to move can rotate it. This combination of forces is called a couple.

(1) A couple is defined as combination of two equal but oppositely directed force not acting along the same line. The effect of couple is known by its moment of couple or torque by a couple $\vec{\tau} = \vec{r} \times \vec{F}$.

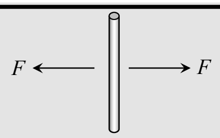
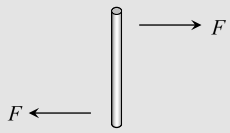
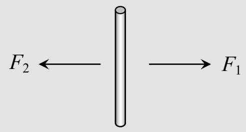
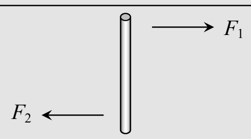


(2) Generally both couple and torque carry equal meaning. The basic difference between torque and couple is the fact that in case of couple both the forces are externally applied while in case of torque one force is externally applied and the other is reactionary.

(3) Work done by torque in twisting the wire $W = \frac{1}{2} C\theta^2$.

Where $\tau = C\theta$; C is known as twisting coefficient or couple per unit twist.

7.16 Translatory and Rotatory Equilibrium

Forces are equal and act along the same line.		$\sum F = 0$ and $\sum \tau = 0$	Body will remain stationary if initially it was at rest.
Forces are equal and does not act along the same line.		$\sum F = 0$ and $\sum \tau \neq 0$	Rotation *i.e.* spinning.
Forces are unequal and act along the same line.		$\sum F \neq 0$ and $\sum \tau = 0$	Translation *i.e.* slipping or skidding.
Forces are unequal and does not act along the same line.		$\sum F \neq 0$ and $\sum \tau \neq 0$	Rotation and translation both *i.e.* rolling.

Sample problems based on torque and couple

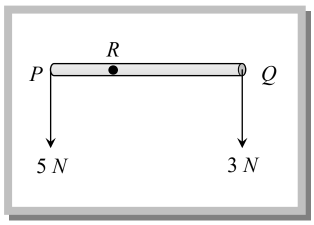
Problem 29. A force of $(2\hat{i} - 4\hat{j} + 2\hat{k}) N$ acts at a point $(3\hat{i} + 2\hat{j} - 4\hat{k})$ *metre* from the origin. The magnitude of torque is

(a) Zero (b) 24.4 *N-m* (c) 0.244 *N-m* (d) 2.444 *N-m*

Solution: (b) $\vec{F} = (2\hat{i} - 4\hat{j} + 2\hat{k})\, N$ and $\vec{r} = (3\hat{i} + 2 - 4\hat{k})$ *meter*

Torque $\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -4 \\ 2 & -4 & 2 \end{vmatrix} \Rightarrow \vec{\tau} = -12\hat{i} - 14\hat{j} - 16\hat{k}$ and $|\vec{\tau}| = \sqrt{(-12)^2 + (-14)^2 + (-16)^2} = 24.4$ *N-m*

Problem 30. The resultant of the system in the figure is a force of $8N$ parallel to the given force through R. The value of PR equals to



(a) $1/4\ RQ$

(b) $3/8\ RQ$

(c) $3/5\ RQ$

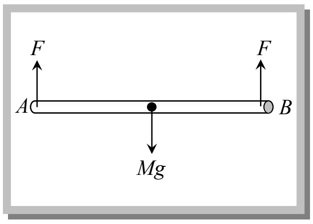
(d) $2/5\ RQ$

Solution: (c) By taking moment of forces about point *R*, $5 \times PR - 3 \times RQ = 0 \Rightarrow PR = \frac{3}{5} RQ$.

Problem 31. A horizontal heavy uniform bar of weight *W* is supported at its ends by two men. At the instant, one of the men lets go off his end of the rod, the other feels the force on his hand changed to

(a) W (b) $\frac{W}{2}$ (c) $\frac{3W}{4}$ (d) $\frac{W}{4}$

Solution: (d) Let the mass of the rod is M $\therefore$ Weight (W) = Mg

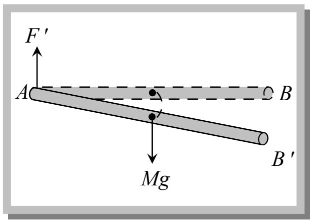


Initially for the equilibrium $F + F = Mg \Rightarrow F = Mg/2$

When one man withdraws, the torque on the rod

$$\tau = I\alpha = Mg\frac{l}{2}$$

$$\Rightarrow \frac{Ml^2}{3}\alpha = Mg\frac{l}{2} \qquad [\text{As } I = Ml^2/3]$$



$\Rightarrow$ Angular acceleration $\alpha = \frac{3}{2}\frac{g}{l}$

and linear acceleration $a = \frac{l}{2}\alpha = \frac{3g}{4}$

Now if the new normal force at A is F then $Mg - F = Ma$

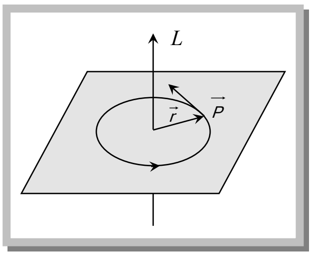
$$\Rightarrow F' = Mg - Ma = Mg - \frac{3Mg}{4} = \frac{Mg}{4} = \frac{W}{4}.$$

7.17 Angular Momentum

The turning momentum of particle about the axis of rotation is called the angular momentum of the particle.

or

The moment of linear momentum of a body with respect to any axis of rotation is known as angular momentum. If $\vec{P}$ is the linear momentum of particle and $\vec{r}$ its position vector from the point of rotation then angular momentum.



$$\vec{L} = \vec{r} \times \vec{P}$$

$$\vec{L} = r\,P\sin\phi\,\hat{n}$$

Angular momentum is an axial vector *i.e.* always directed perpendicular to the plane of rotation and along the axis of rotation.

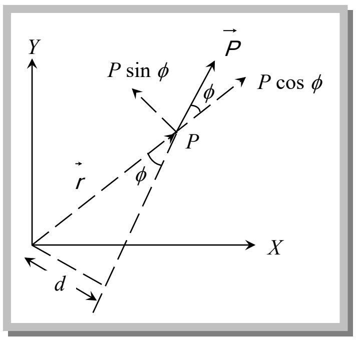
(1) S.I. Unit : $kg\text{-}m^2\text{-}s^{-1}$ or *J-sec*.

(2) Dimension : $[ML^2T^{-1}]$ and it is similar to Planck's constant (h).

(3) In cartesian co-ordinates if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

Then $\vec{L} = \vec{r} \times \vec{P} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} = (yP_z - zP_y)\hat{i} - (xP_z - zP_x)\hat{j} + (xP_y - yP_x)\hat{k}$

(4) As it is clear from the figure radial component of momentum $\vec{P_r} = P\cos\phi$



Transverse component of momentum $\vec{P_\phi} = P\sin\phi$

So magnitude of angular momentum $L = r\,P\sin\phi$

$$L = r P_\phi$$

$\therefore$ Angular momentum = Position vector × Transverse component of angular momentum

i.e., The radial component of linear momentum has no role to play in angular momentum.

(5) Magnitude of angular momentum $L = P \ (r \sin\phi) = L = Pd$ [As $d = r\sin\phi$ from the figure.]

$\therefore$ Angular momentum = (Linear momentum) × (Perpendicular distance of line of action of force from the axis of rotation)

(6) Maximum and minimum angular momentum : We know $\vec{L} = \vec{r} \times \vec{P}$

$\therefore$ $\vec{L} = m[\vec{r} \times \vec{v}] = mvr\sin\phi = Pr\sin\phi$ [As $\vec{P} = m\vec{v}$]

$L_{maximum} = mvr$	When $\lvert \sin\phi \rvert = \max = 1$ *i.e.*, $\phi = 90^\circ$	$\vec{v}$ is perpendicular to $\vec{r}$
$L_{minimum} = 0$	When $\lvert \sin\phi \rvert = \min = 0$ *i.e.* $\phi = 0^\circ$ or 180°	$\vec{v}$ is parallel or anti-parallel to $\vec{r}$

(7) A particle in translatory motion always have an angular momentum unless it is a point on the line of motion because $L = mvr\sin\phi$ and $L > 1$ if $\phi \neq 0^o$ or 180^o

(8) In case of circular motion, $\vec{L} = \vec{r} \times \vec{P} = m(\vec{r} \times \vec{v}) = mvr\sin\phi$

$\therefore$ $L = mvr = mr^2\omega$ [As $\vec{r} \perp \vec{v}$ and $v = r\omega$]

or $L = I\omega$ [As $mr^2 = I$]

In vector form $\vec{L} = I\vec{\omega}$

(9) From $\vec{L} = I\vec{\omega}$ $\therefore$ $\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} = \vec{\tau}$ [As $\frac{d\vec{\omega}}{dt} = \vec{\alpha}$ and $\vec{\tau} = I\vec{\alpha}$]

i.e. the rate of change of angular momentum is equal to the net torque acting on the particle. [Rotational analogue of Newton's second law]

(10) If a large torque acts on a particle for a small time then 'angular impulse' of torque is given by

$$\vec{J} = \int \vec{\tau}\, dt = \vec{\tau}_{av} \int_{t_1}^{t_2} dt$$

or Angular impulse $\vec{J} = \vec{\tau}_{av} \Delta t = \Delta \vec{L}$

$\therefore$ Angular impulse = Change in angular momentum

(11) The angular momentum of a system of particles is equal to the vector sum of angular momentum of each particle *i.e.*, $\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + ... + \vec{L}_n$.

(12) According to Bohr theory angular momentum of an electron in n^{th} orbit of atom can be taken as,

$$L = n\frac{h}{2\pi}$$ [where *n* is an integer used for number of orbit]

7.18 Law of Conservation of Angular Momentum

Newton's second law for rotational motion $\vec{\tau} = \frac{d\vec{L}}{dt}$

So if the net external torque on a particle (or system) is zero then $\frac{d\vec{L}}{dt} = 0$

i.e. $\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \ldots\ldots = \text{constant.}$

Angular momentum of a system (may be particle or body) remains constant if resultant torque acting on it zero.

As $L = I\omega$ so if $\vec{\tau} = 0$ then $I\omega = \text{constant}$ $\therefore$ $I \propto \frac{1}{\omega}$

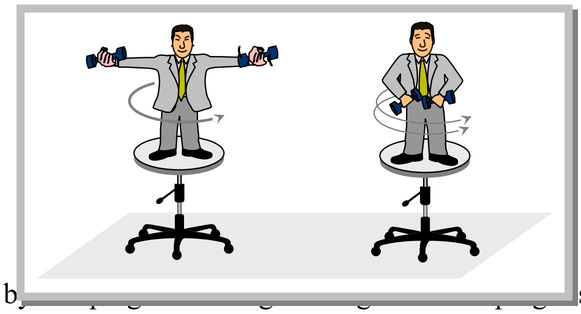
Since angular momentum $I\omega$ remains constant so when I decreases, angular velocity ω increases and vice-versa.

Examples of law of conservation of angular momentum :

(1) The angular velocity of revolution of a planet around the sun in an elliptical orbit increases when the planet come closer to the sun and vice-versa because when planet comes closer to the sun, it's moment of inertia I decreases there fore ω increases.

(2) A circus acrobat performs feats involving spin by bringing his arms and legs closer to his body or vice-versa. On bringing the arms and legs closer to body, his moment of inertia I decreases. Hence ω increases.

(3) A person-carrying heavy weight in his hands and standing on a rotating platform can change the speed of platform. When the person suddenly folds his arms. Its moment of inertia decreases and in accordance the angular speed increases.



(4) A diver performs somersaults b[illegible]s legs and arms out stretched first and then curling his body.

(5) Effect of change in radius of earth on its time period

Angular momentum of the earth $L = I\omega = \text{constant}$

$$L = \frac{2}{5}MR^2 \times \frac{2\pi}{T} = \text{constant}$$

$\therefore$ $T \propto R^2$ [if M remains constant]

If R becomes half then time period will become one-fourth *i.e.* $\frac{24}{4} = 6\,hrs.$

Sample problems based on angular momentum

***Problem* 32.** Consider a body, shown in figure, consisting of two identical balls, each of mass M connected by a light rigid rod. If an impulse $J = Mv$ is imparted to the body at one of its ends, what would be its angular velocity

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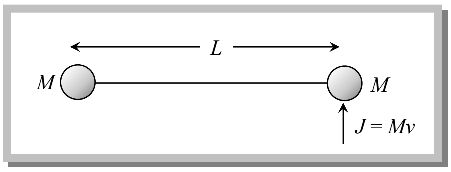


(a) v/L

(b) $2v/L$

(c) $v/3L$

(d) $v/4L$

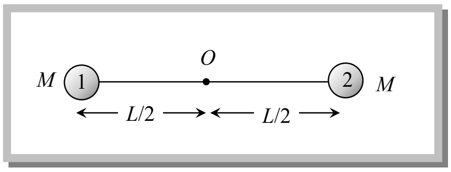
Solution: (a) Initial angular momentum of the system about point O

= Linear momentum × Perpendicular distance of linear momentum from the axis of rotation $= Mv\left(\frac{L}{2}\right)$ ....(i)

Final angular momentum of the system about point $O = I_1\omega + I_2\omega = (I_1 + I_2)\omega$

$= \left[M\left(\frac{L}{2}\right)^2 + M\left(\frac{L}{2}\right)^2\right]\omega$ ....(ii)

Applying the law of conservation of angular momentum



$\Rightarrow \quad Mv\left(\frac{L}{2}\right) = 2M\left(\frac{L}{2}\right)^2\omega$

$\Rightarrow \quad \omega = \frac{v}{L}$

***Problem* 33.** A thin circular ring of mass M and radius R is rotating about its axis with a constant angular velocity ω. Four objects each of mass m, are kept gently to the opposite ends of two perpendicular diameters of the ring. The angular velocity of the ring will be

(a) $\frac{M\omega}{M+4m}$ (b) $\frac{(M+4m)\omega}{M}$ (c) $\frac{(M-4m)\omega}{M+4m}$ (d) $\frac{M\omega}{4m}$

Solution: (a) Initial angular momentum of ring $= I\omega = MR^2\omega$

If four object each of mass m, and kept gently to the opposite ends of two perpendicular diameters of the ring then final angular momentum = $(MR^2 + 4mR^2)\omega'$

By the conservation of angular momentum

Initial angular momentum = Final angular momentum

$MR^2\omega = (MR^2 + 4mR^2)\omega' \Rightarrow \omega' = \left(\frac{M}{M+4m}\right)\omega$.