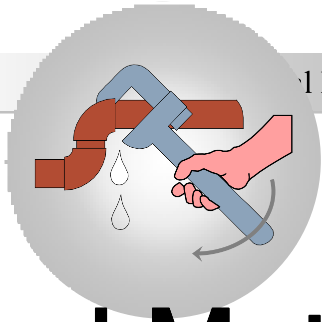


Rotational Motion

7.1 Introduction

Translation is motion along a straight line but rotation is the motion of wheels, gears, motors, planets, the hands of a clock, the rotor of jet engines and the blades of helicopters. First figure shows a skater gliding across the ice in a straight line with constant speed. Her motion is called translation but second figure shows her spinning at a constant rate about a vertical axis. Here motion is called rotation.

Up to now we have studied translatory motion of a point mass. In this chapter we will study the rotatory motion of rigid body about a fixed axis.

(1) Rigid body : A rigid body is a body that can rotate with all the parts locked together and without any change in its shape.

(2) System : A collection of any number of particles interacting with one another and are under consideration during analysis of a situation are said to form a system.

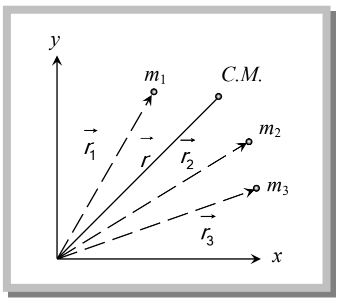
(3) Internal forces : All the forces exerted by various particles of the system on one another are called internal forces. These forces are alone enable the particles to form a well defined system. Internal forces between two particles are mutual (equal and opposite).

(4) External forces : To move or stop an object of finite size, we have to apply a force on the object from outside. This force exerted on a given system is called an external force.

7.2 Centre of Mass

Centre of mass of a system (body) is a point that moves as though all the mass were concentrated there and all external forces were applied there.

(1) **Position vector of centre of mass for *n* particle system :** If a system consists of *n* particles of masses $m_1, m_2, m_3......m_n$, whose positions vectors are $\vec{r}_1, \vec{r}_2, \vec{r}_3 .... \vec{r}_n$ respectively then position vector of centre of mass



$$\vec{r} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + ... m_n \vec{r}_n}{m_1 + m_2 + m_3 + ... m_n}$$

Hence the centre of mass of *n* particles is a weighted average of the position vectors of *n* particles making up the system.

(2) **Position vector of centre of mass for two particle system :** $\vec{r} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$

and the centre of mass lies between the particles on the line joining them.

If two masses are equal *i.e.* $m_1 = m_2$, then position vector of centre of mass $\vec{r} = \frac{\vec{r_1} + \vec{r_2}}{2}$

(3) **Important points about centre of mass**

(i) The position of centre of mass is independent of the co-ordinate system chosen.

(ii) The position of centre of mass depends upon the shape of the body and distribution of mass.

Example : The centre of mass of a circular disc is within the material of the body while that of a circular ring is outside the material of the body.

(iii) In symmetrical bodies in which the distribution of mass is homogenous, the centre of mass coincides with the geometrical centre or centre of symmetry of the body.

(iv) Position of centre of mass for different bodies

S. No.	Body	Position of centre of mass
(a)	Uniform hollow sphere	Centre of sphere
(b)	Uniform solid sphere	Centre of sphere
(c)	Uniform circular ring	Centre of ring
(d)	Uniform circular disc	Centre of disc
(e)	Uniform rod	Centre of rod
(f)	A plane lamina (Square, Rectangle, Parallelogram)	Point of inter section of diagonals
(g)	Triangular plane lamina	Point of inter section of medians
(h)	Rectangular or cubical block	Points of inter section of diagonals
(i)	Hollow cylinder	Middle point of the axis of cylinder
(j)	Solid cylinder	Middle point of the axis of cylinder
(k)	Cone or pyramid	On the axis of the cone at point distance $\frac{3h}{4}$ from the vertex where *h* is the height of cone

(v) The centre of mass changes its position only under the translatory motion. There is no effect of rotatory motion on centre of mass of the body.

(vi) If the origin is at the centre of mass, then the sum of the moments of the masses of the system about the centre of mass is zero *i.e.* $\Sigma m_i \vec{r_i} = 0$.

(vii) If a system of particles of masses $m_1, m_2, m_3, ...$ move with velocities $v_1, v_2, v_3, ...$

then the velocity of centre of mass $v_{cm} = \frac{\Sigma m_i v_i}{\Sigma m_i}$.

(viii) If a system of particles of masses $m_1, m_2, m_3, ...$ move with accelerations $a_1, a_2, a_3, ...$

then the acceleration of centre of mass $A_{cm} = \frac{\sum m_i a_i}{\sum m_i}$

(ix) If $\vec{r}$ is a position vector of centre of mass of a system

then velocity of centre of mass $\vec{v}_{cm} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...}{m_1 + m_2 + m_3 + ...}\right)$

(x) Acceleration of centre of mass $\vec{A}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2}{dt^2}\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ...}{m_1 + m_2 + m_3 + ...}\right)$

(xi) Force on a rigid body $\vec{F} = M\vec{A}_{cm} = M\frac{d^2\vec{r}}{dt^2}$

(xii) For an isolated system external force on the body is zero

$$\vec{F} = M\frac{d}{dt}\left(\vec{v}_{cm}\right) = 0 \Rightarrow \vec{v}_{cm} = \text{constant.}$$

i.e., centre of mass of an isolated system moves with uniform velocity along a straight-line path.

Sample problems based on centre of mass

***Problem* 1.** The distance between the carbon atom and the oxygen atom in a carbon monoxide molecule is 1.1 Å. Given, mass of carbon atom is 12 *a.m.u.* and mass of oxygen atom is 16 *a.m.u.*, calculate the position of the center of mass of the carbon monoxide molecule

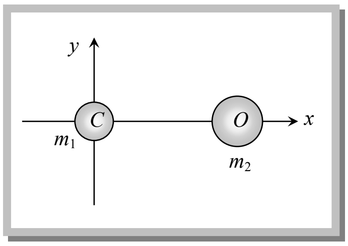
(a) 6.3 Å from the carbon atom
(b) 1 Å from the oxygen atom
(c) 0.63 Å from the carbon atom
(d) 0.12 Å from the oxygen atom

Solution : (c) Let carbon atom is at the origin and the oxygen atom is placed at *x*-axis

$m_1 = 12,\ m_2 = 16,\ \vec{r}_1 = 0\hat{i} + 0\hat{j}$ and $\vec{r}_2 = 1.1\hat{i} + 0\hat{j}$

$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{16 \times 1.1}{28}\hat{i}$

$\vec{r} = 0.63\hat{i}$ *i.e.* 0.63 Å from carbon atom.



***Problem* 2.** The velocities of three particles of masses 20*g*, 30*g* and 50 *g* are $10\vec{i}, 10\vec{j}$, and $10\vec{k}$ respectively. The velocity of the centre of mass of the three particles is **[EAMCET 2001]**

(a) $2\vec{i} + 3\vec{j} + 5\vec{k}$
(b) $10(\vec{i} + \vec{j} + \vec{k})$
(c) $20\vec{i} + 30\vec{j} + 5\vec{k}$
(d) $2\vec{i} + 30\vec{j} + 50\vec{k}$

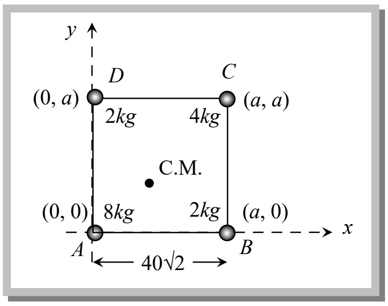
Solution : (a) Velocity of centre of mass $v_{cm} = \frac{m_1v_1 + m_2v_2 + m_3v_3}{m_1 + m_2 + m_3} = \frac{20 \times 10\hat{i} + 30 \times 10\hat{j} + 50 \times 10\hat{k}}{100} = 2\hat{i} + 3\hat{j} + 5\hat{k}$.

***Problem* 3.** Masses $8, 2, 4, 2\ kg$ are placed at the corners A, B, C, D respectively of a square $ABCD$ of diagonal $80\ cm$. The distance of centre of mass from A will be

(a) $20\ cm$
(b) $30\ cm$
(c) $40\ cm$
(d) $60\ cm$

Solution : (b) Let corner A of square *ABCD* is at the origin and the mass 8 *kg* is placed at this corner (given in problem) Diagonal of square $d = a\sqrt{2} = 80\,cm \Rightarrow a = 40\sqrt{2}cm$

$m_1 = 8kg,\ m_2 = 2kg,\ m_3 = 4kg,\ m_4 = 2kg$



Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$ are the position vectors of respective masses

$\vec{r}_1 = 0\hat{i} + 0\hat{j},\ \vec{r}_2 = a\hat{i} + 0\hat{j},\ \vec{r}_3 = a\hat{i} + a\hat{j},\ \vec{r}_4 = 0\hat{i} + a\hat{j}$

From the formula of centre of mass

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4}{m_1 + m_2 + m_3 + m_4} = 15\sqrt{2}\hat{i} + 15\sqrt{2}\hat{j}$$

$\therefore$ co-ordinates of centre of mass $= (15\sqrt{2}, 15\sqrt{2})$ and co-ordination of the corner $= (0, 0)$

From the formula of distance between two points (x_1, y_1) and (x_2, y_2)

distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(15\sqrt{2} - 0)^2 + (15\sqrt{2} - 0)^2} = \sqrt{900} = 30cm$

***Problem* 4.** The coordinates of the positions of particles of mass $7, 4$ and $10\,gm$ are $(1, 5, -3), (2, 5, 7)$ and $(3, 3, -1)\,cm$ respectively. The position of the centre of mass of the system would be

(a) $\left(-\frac{15}{7}, \frac{85}{17}, \frac{1}{7}\right)cm$ (b) $\left(\frac{15}{7}, -\frac{85}{17}, \frac{1}{7}\right)cm$ (c) $\left(\frac{15}{7}, \frac{85}{21}, -\frac{1}{7}\right)cm$ (d) $\left(\frac{15}{7}, \frac{85}{21}, \frac{7}{3}\right)cm$

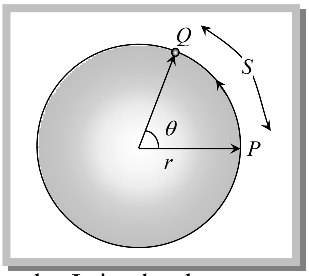
Solution: (c) $m_1 = 7gm,\ m_2 = 4gm,\ m_3 = 10gm$ and $\vec{r}_1 = (\hat{i} + 5\hat{j} - 3\hat{k}),\ \vec{r}_2 = (2\hat{i} + 5\hat{j} + 7\hat{k}),\ \vec{r}_3 = (3\hat{i} + 3\hat{j} - \hat{k})$

Position vector of center mass $\vec{r} = \frac{7(\hat{i} + 5\hat{j} - 3\hat{k}) + 4(2\hat{i} + 5\hat{j} + 7\hat{k}) + 10(3\hat{i} + 3\hat{j} - \hat{k})}{7 + 4 + 10} = \frac{(45\hat{i} + 85\hat{j} - 3\hat{k})}{21}$

$\Rightarrow \vec{r} = \frac{15}{7}\hat{i} + \frac{85}{21}\hat{j} - \frac{1}{7}\hat{k}$. So coordinates of centre of mass $\left[\frac{15}{7}, \frac{85}{21}, \frac{-1}{7}\right]$.

7.3 Angular Displacement

It is the angle described by the position vector $\vec{r}$ about the axis of rotation.



Angular displacement $(\theta) = \frac{\text{Linear displacement } (s)}{\text{Radius}(r)}$

(1) Unit : radian

(2) Dimension : $[M^0 L^0 T^0]$

(3) Vector form $\vec{S} = \vec{\theta} \times \vec{r}$

i.e., angular displacement is a vector quantity whose direction is given by right hand rule. It is also known as axial vector. For anti-clockwise sense of rotation direction of θ is perpendicular to the plane, outward and along the axis of rotation and vice-versa.

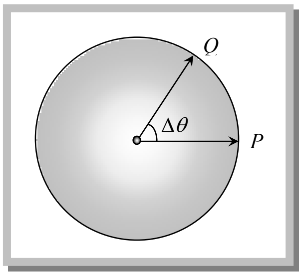
(4) 2π radian $= 360° = 1$ revolution.

(5) If a body rotates about a fixed axis then all the particles will have same angular displacement (although linear displacement will differ from particle to particle in accordance with the distance of particles from the axis of rotation).

7.4 Angular Velocity

The angular displacement per unit time is defined as angular velocity.

If a particle moves from *P* to *Q* in time Δt, $\omega = \frac{\Delta\theta}{\Delta t}$ where $\Delta\theta$ is the angular displacement.



(1) Instantaneous angular velocity $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

(2) Average angular velocity $\omega_{av} = \frac{\text{total angular displacement}}{\text{total time}} = \frac{\theta_2 - \theta_1}{t_2 - t_1}$

(3) Unit : *Radian/sec*

(4) Dimension : $[M^0 L^0 T^{-1}]$ which is same as that of frequency.

(5) Vector form $\vec{v} = \vec{\omega} \times \vec{r}$ [where $\vec{v}$ = linear velocity, $\vec{r}$ = radius vector]

$\vec{\omega}$ is a axial vector, whose direction is normal to the rotational plane and its direction is given by right hand screw rule.

(6) $\omega = \frac{2\pi}{T} = 2\pi n$ [where T = time period, n = frequency]

(7) The magnitude of an angular velocity is called the angular speed which is also represented by ω.

7.5 Angular Acceleration

The rate of change of angular velocity is defined as angular acceleration.

If particle has angular velocity ω_1 at time t_1 and angular velocity ω_2 at time t_2 then,

Angular acceleration $\vec{\alpha} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1}$

(1) Instantaneous angular acceleration $\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2}$.

(2) Unit : rad/sec^2

(3) Dimension : $[M^0 L^0 T^{-2}]$.

(4) If $\alpha = 0$, circular or rotational motion is said to be uniform.

(5) Average angular acceleration $\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$.

(6) Relation between angular acceleration and linear acceleration $\vec{a} = \vec{\alpha} \times \vec{r}$.

(7) It is an axial vector whose direction is along the change in direction of angular velocity *i.e.* normal to the rotational plane, outward or inward along the axis of rotation (depends upon the sense of rotation).

7.6 Equations of Linear Motion and Rotational Motion

	Linear Motion	Rotational Motion
(1)	If linear acceleration is 0, u = constant and $s = u\,t$.	If angular acceleration is 0, ω = constant and $\theta = \omega t$
(2)	If linear acceleration a = constant, (i) $s = \frac{(u+v)}{2} t$	If angular acceleration α = constant then (i) $\theta = \frac{(\omega_1 + \omega_2)}{2} t$

	(ii) $a = \frac{v-u}{t}$	(ii) $\alpha = \frac{\omega_2 - \omega_1}{t}$
	(iii) $v = u + at$	(iii) $\omega_2 = \omega_1 + \alpha t$
	(iv) $s = ut + \frac{1}{2}at^2$	(iv) $\theta = \omega_1 t + \frac{1}{2}\alpha t^2$
	(v) $v^2 = u^2 + 2as$	(v) $\omega_2^2 = \omega_1^2 + 2\alpha\theta$
	(vi) $s_{nth} = u + \frac{1}{2}a(2n-1)$	(vi) $\theta_{nth} = \omega_1 + (2n-1)\frac{\alpha}{2}$
(3)	If acceleration is not constant, the above equation will not be applicable. In this case (i) $v = \frac{dx}{dt}$ (ii) $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ (iii) $vdv = a\,ds$	If acceleration is not constant, the above equation will not be applicable. In this case (i) $\omega = \frac{d\theta}{dt}$ (ii) $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$ (iii) $\omega d\omega = \alpha\, d\theta$

Sample problems based on angular displacement, velocity and acceleration

***Problem* 5.** The angular velocity of *seconds* hand of a watch will be

(a) $\frac{\pi}{60}\,rad/sec$ (b) $\frac{\pi}{30}\,rad/sec$ (c) $60\pi\,rad/sec$ (d) $30\pi\,rad/sec$

Solution : (b) We know that second's hand completes its revolution (2π) in 60 *sec* $\therefore\ \omega = \frac{\theta}{t} = \frac{2\pi}{60} = \frac{\pi}{30}\,rad/sec$

***Problem* 6.** The wheel of a car is rotating at the rate of 1200 revolutions per *minute*. On pressing the accelerator for 10 *sec* it starts rotating at 4500 revolutions per *minute*. The angular acceleration of the wheel is **[MP PET 2001]**

(a) 30 *radians/sec*² (b) 1880 *degrees/sec*² (c) 40 *radians/sec*² (d) 1980 *degrees/sec*²

Solution: (d) Angular acceleration (α) = rate of change of angular speed

$$= \frac{2\pi(n_2 - n_1)}{t} = \frac{2\pi\left(\frac{4500-1200}{60}\right)}{10} = \frac{2\pi\frac{3300}{60}}{10} \times \frac{360}{2\pi}\frac{degree}{sec^2} = 1980\ degree/sec^2.$$

***Problem* 7.** Angular displacement (θ) of a flywheel varies with time as $\theta = at + bt^2 + ct^3$ then angular acceleration is given by

(a) $a + 2bt - 3ct^2$ (b) $2b - 6t$ (c) $a + 2b - 6t$ (d) $2b + 6ct$

Solution: (d) Angular acceleration $\alpha = \frac{d^2\theta}{dt^2} = \frac{d^2}{dt^2}(at + bt^2 + ct^3) = 2b + 6ct$

***Problem* 8.** A wheel completes 2000 rotations in covering a distance of $9.5\,km$. The diameter of the wheel is **[RPMT**

(a) $1.5\,m$ (b) $1.5\,cm$ (c) $7.5\,m$ (d) $7.5\,cm$

Solution: (a) Distance covered by wheel in 1 rotation = $2\pi r = \pi D$ (Where $D = 2r$ = diameter of wheel)

$\therefore$ Distance covered in 2000 rotation = 2000 $\pi D = 9.5 \times 10^3\,m$ (given)

$\therefore D = 1.5\ meter$

***Problem* 9.** A wheel is at rest. Its angular velocity increases uniformly and becomes 60 *rad/sec* after 5 *sec*. The total angular displacement is

(a) $600\ rad$ (b) $75\ rad$ (c) $300\ rad$ (d) $150\ rad$

Solution: (d) Angular acceleration $\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{60-0}{5} = 12 rad / sec^2$

Now from $\theta = \omega_1 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}(12)(5)^2 = 150\ rad.$

***Problem* 10.** A wheel initially at rest, is rotated with a uniform angular acceleration. The wheel rotates through an angle θ_1 in first one second and through an additional angle θ_2 in the next one second. The ratio $\frac{\theta_2}{\theta_1}$ is

(a) 4 (b) 2 (c) 3 (d) 1

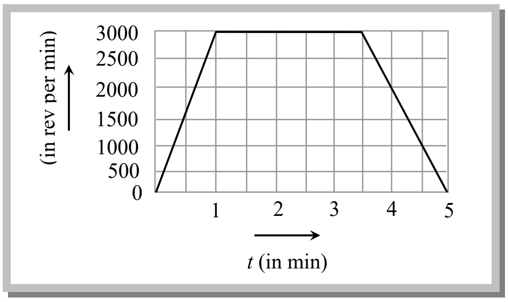
Solution: (c) Angular displacement in first one second $\theta_1 = \frac{1}{2}\alpha(1)^2 = \frac{\alpha}{2}$ ......(i) [From $\theta = \omega_1 t + \frac{1}{2}\alpha t^2$]

Now again we will consider motion from the rest and angular displacement in total two seconds

$\theta_1 + \theta_2 = \frac{1}{2}\alpha(2)^2 = 2\alpha$ ......(ii)

Solving (i) and (ii) we get $\theta_1 = \frac{\alpha}{2}$ and $\theta_2 = \frac{3\alpha}{2}$ $\therefore \frac{\theta_2}{\theta_1} = 3$.

***Problem* 11.** As a part of a maintenance inspection the compressor of a jet engine is made to spin according to the graph as shown. The number of revolutions made by the compressor during the test is

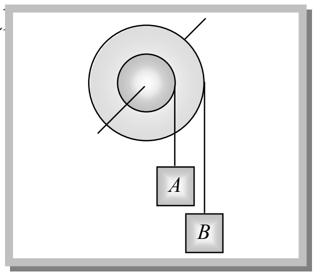


(a) 9000 (b) 16570 (c) 12750 (d) 11250

Solution: (d) Number of revolution = Area between the graph and time axis = Area of trapezium

$= \frac{1}{2} \times (2.5 + 5) \times 3000 =$ 11250 revolution.

***Problem* 12.** Figure shows a small wheel fixed coaxially on a bigger one of double the radius. The system rotates about the common axis. The strings supporting *A* and *B* do not slip on the wheels. If *x* and *y* be the distances travelled by *A* and *B* in the same time interva



(a) $x = 2y$

(b) $x = y$

(c) $y = 2x$

(d) None of these

Solution: (c) Linear displacement (S) = Radius (r) × Angular displacement (θ)

$\therefore S \propto r$ (if $\theta =$ constant)

$$\frac{\text{Distance travelled by mass } A(x)}{\text{Distance travelled by mass } B(y)} = \frac{\text{Radius of pulley concerned with mass } A(r)}{\text{Radius of pulley concerned with mass } B(2r)} = \frac{1}{2} \Rightarrow y = 2x.$$

Problem **13.** If the position vector of a particle is $\vec{r} = (3\hat{i} + 4\hat{j})$ meter and its angular velocity is $\vec{\omega} = (\hat{j} + 2\hat{k})$ *rad*/*sec* then its linear velocity is (in *m*/*s*)

(a) $(8\hat{i} - 6\hat{j} + 3\hat{k})$ (b) $(3\hat{i} + 6\hat{j} + 8\hat{k})$ (c) $-(3\hat{i} + 6\hat{j} + 6\hat{k})$ (d) $(6\hat{i} + 8\hat{j} + 3\hat{k})$

Solution: (a) $\vec{v} = \vec{\omega} \times \vec{r} = (3\hat{i} + 4\hat{j} + 0\hat{k}) \times (0\hat{i} + \hat{j} + 2\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 8\hat{i} - 6\hat{j} + 3\hat{k}$

7.7 Moment of Inertia

Moment of inertia plays the same role in rotational motion as mass plays in linear motion. It is the property of a body due to which it opposes any change in its state of rest or of uniform rotation.

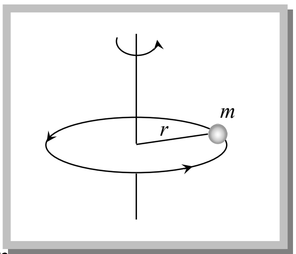
(1) Moment of inertia of a particle $I = mr^2$; where r is the perpendicular distance of particle from rotational axis.

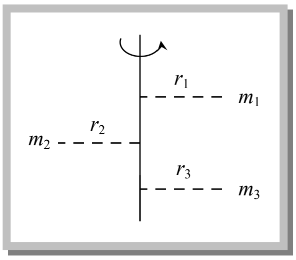
(2) Moment of inertia of a body made up of number of particles (discrete distribution)

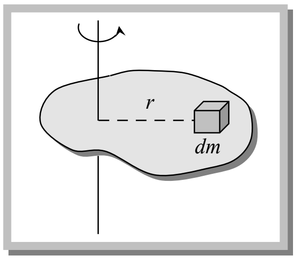
$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...$$

(3) Moment of inertia of a continuous distribution of mass, treating the element of mass dm at position r as particle

$$dI = dm r^2 \quad \textit{i.e.}, \quad I = \int r^2 dm$$







(4) Dimension : $[ML^2T^0]$

(5) S.I. unit : kgm^2.

(6) Moment of inertia depends on mass, distribution of mass and on the position of axis of rotation.

(7) Moment of inertia does not depend on angular velocity, angular acceleration, torque, angular momentum and rotational kinetic energy.

(8) It is not a vector as direction (clockwise or anti-clockwise) is not to be specified and also not a scalar as it has different values in different directions. Actually it is a tensor quantity.

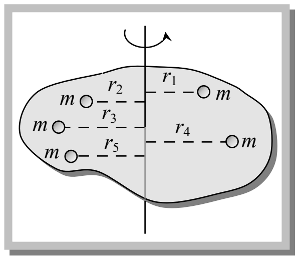
(9) In case of a hollow and solid body of same mass, radius and shape for a given axis, moment of inertia of hollow body is greater than that for the solid body because it depends upon the mass distribution.

7.8 Radius of Gyration

Radius of gyration of a body about a given axis is the perpendicular distance of a point from the axis, where if whole mass of the body were concentrated, the body shall have the same moment of inertia as it has with the actual distribution of mass.

When square of radius of gyration is multiplied with the mass of the body gives the moment of inertia of the body about the given axis.

$$I = Mk^2 \text{ or } k = \sqrt{\frac{I}{M}}.$$



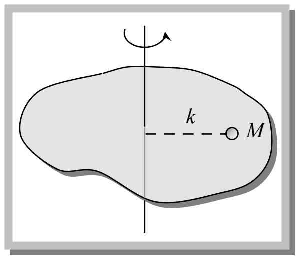
Here k is called radius of gyration.

From the formula of discrete distribution

$$I = mr_1^2 + mr_2^2 + mr_3^2 + ... + mr_n^2$$

If $m_1 = m_2 = m_3 = ... = m$ then

$$I = m(r_1^2 + r_2^2 + r_3^2 + ...r_n^2) \qquad ....(i)$$



From the definition of Radius of gyration,

$$I = Mk^2 \qquad ....(ii)$$

By equating (i) and (ii)

$$Mk^2 = m(r_1^2 + r_2^2 + r_3^2 + ... + r_n^2)$$

$$nmk^2 = m(r_1^2 + r_2^2 + r_3^2 + ... + r_n^2) \qquad [\text{As } M = nm]$$

$$\therefore \qquad k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + ... + r_n^2}{n}}$$

Hence radius of gyration of a body about a given axis is equal to root mean square distance of the constituent particles of the body from the given axis.

(1) Radius of gyration (k) depends on shape and size of the body, position and configuration of the axis of rotation, distribution of mass of the body *w.r.t.* the axis of rotation.

(2) Radius of gyration (k) does not depends on the mass of body.

(3) Dimension $[M^0 L^1 T^0]$.

(4) S.I. unit : *Meter*.

(5) Significance of radius of gyration : Through this concept a real body (particularly irregular) is replaced by a point mass for dealing its rotational motion.

Example : In case of a disc rotating about an axis through its centre of mass and perpendicular to its plane

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{(1/2)MR^2}{M}} = \frac{R}{\sqrt{2}}$$

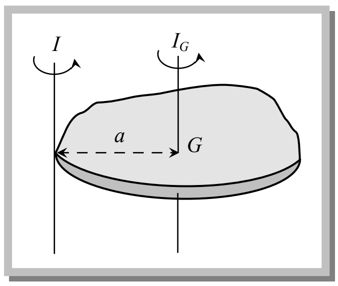
So instead of disc we can assume a point mass M at a distance $(R/\sqrt{2})$ from the axis of rotation for dealing the rotational motion of the disc.

Note : ❑ For a given body inertia is constant whereas moment of inertia is variable.

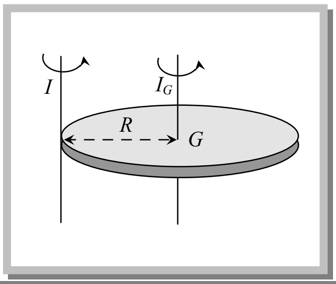
7.9 Theorem of Parallel Axes

Moment of inertia of a body about a given axis I is equal to the sum of moment of inertia of the body about an axis parallel to given axis and passing through centre of mass of the body I_g and Ma^2 where M is the mass of the body and a is the perpendicular distance between the two axes.

$$I = I_g + Ma^2$$



Example: Moment of inertia of a disc about an axis through its centre and perpendicular to the plane is $\frac{1}{2}MR^2$, so moment of inertia about an axis through its tangent and perpendicular to the plane will be



$$I = I_g + Ma^2$$

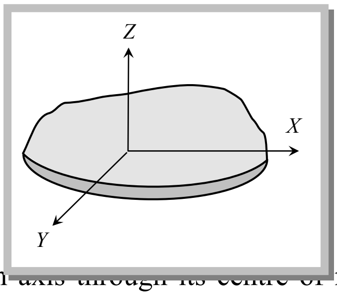
$$I = \frac{1}{2}MR^2 + MR^2$$

$$\therefore \quad I = \frac{3}{2}MR^2$$

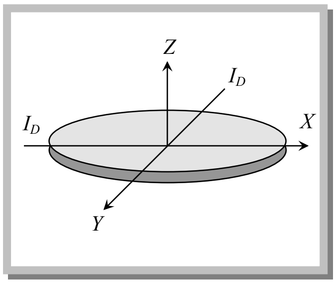
7.10 Theorem of Perpendicular Axes

According to this theorem the sum of moment of inertia of a plane lamina about two mutually perpendicular axes lying in its plane is equal to its moment of inertia about an axis perpendicular to the plane of lamina and passing through the point of intersection of first two axes.

$$I_z = I_x + I_y$$



Example : Moment of inertia of a disc about an axis through its centre of mass and perpendicular to its plane is $\frac{1}{2}MR^2$, so if the disc is in *x–y* plane then by theorem of perpendicular axes



i.e. $\quad I_z = I_x + I_y$

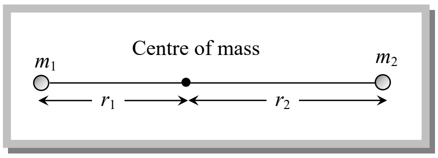
$\Rightarrow \quad \frac{1}{2}MR^2 = 2I_D$ [As ring is symmetrical body so $I_x = I_y = I_D$]

$\Rightarrow \quad I_D = \frac{1}{4}MR^2$

Note : ❑ In case of symmetrical two-dimensional bodies as moment of inertia for all axes passing through the centre of mass and in the plane of body will be same so the two axes in the plane of body need not be perpendicular to each other.

7.11 Moment of Inertia of Two Point Masses About Their Centre of Mass

Let m_1 and m_2 be two masses distant r from each-other and r_1 and r_2 be the distances of their centre of mass from m_1 and m_2 respectively, then



(1) $r_1 + r_2 = r$

(2) $m_1 r_1 = m_2 r_2$

(3) $r_1 = \frac{m_2}{m_1 + m_2} r$ and $\quad r_2 = \frac{m_1}{m_1 + m_2} r$

(4) $I = m_1 r_1^2 + m_2 r_2^2$

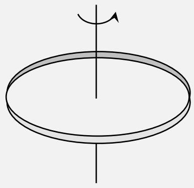
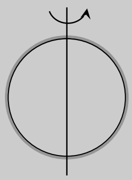
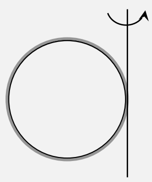
(5) $I = \left[\frac{m_1 m_2}{m_1 + m_2}\right] r^2 = \mu r^2$ [where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is known as reduced mass $\mu < m_1$ and $\mu < m_2$.]

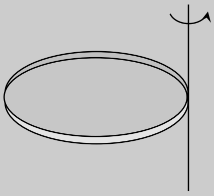
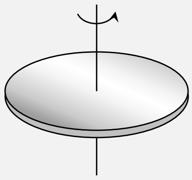
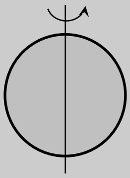
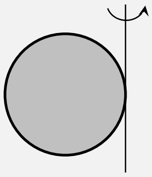
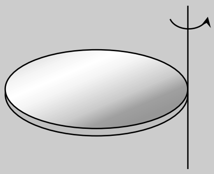
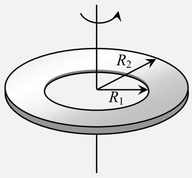
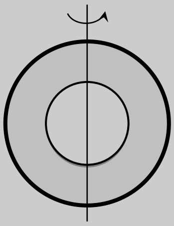
(6) In diatomic molecules like H_2, HCl *etc*. moment of inertia about their centre of mass is derived from above formula.

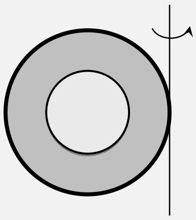
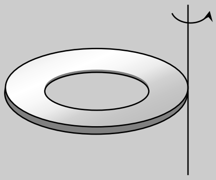
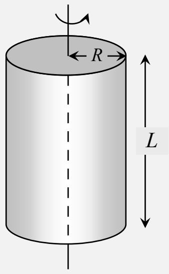
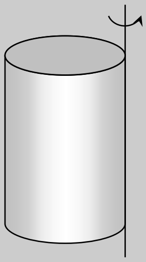
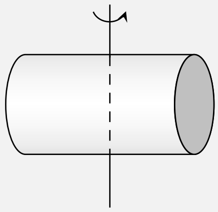
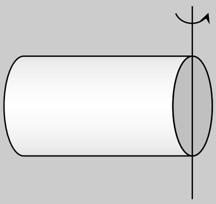
7.12 Analogy Between Tranlatory Motion and Rotational Motion

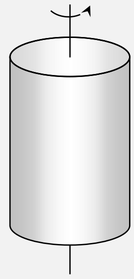
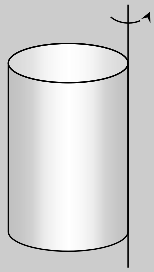
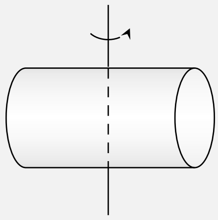
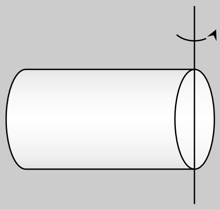
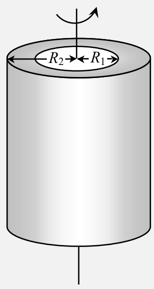
Translatory motion		Rotatory motion	
Mass	(m)	Moment of Inertia	(I)
Linear momentum	$P = mv$ $P = \sqrt{2mE}$	Angular Momentum	$L = I\omega$ $L = \sqrt{2IE}$
Force	$F = ma$	Torque	$\tau = I\alpha$
Kinetic energy	$E = \frac{1}{2}mv^2$ $E = \frac{P^2}{2m}$		$E = \frac{1}{2}I\omega^2$ $E = \frac{L^2}{2I}$

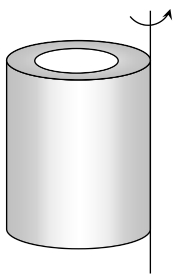
7.13 Moment of Inertia of Some Standard Bodies About Different Axes

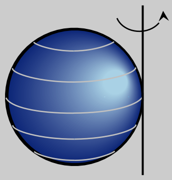
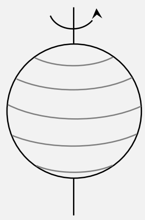
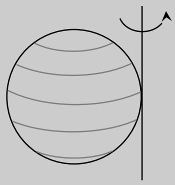
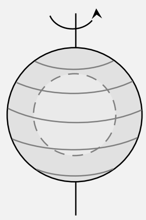
Body	Axis of Rotation	Figure	Moment of inertia	*k*	*k*²/*R*²
Ring	About an axis passing through C.G. and perpendicular to its plane		MR^2	R	1
Ring	About its diameter		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Ring	About a tangential axis in its own plane		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$	$\frac{3}{2}$

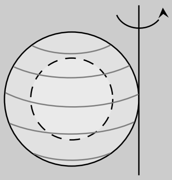
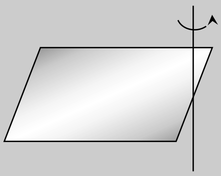
Body	Axis of Rotation	Figure	Moment of inertia	k	k^2/R^2
Ring	About a tangential axis perpendicular to its own plane		$2MR^2$	$\sqrt{2}R$	2
Disc	About an axis passing through C.G. and perpendicular to its plane		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Disc	About its Diameter		$\frac{1}{4}MR^2$	$\frac{R}{2}$	$\frac{1}{4}$
Disc	About a tangential axis in its own plane		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$	$\frac{5}{4}$
Disc	About a tangential axis perpendicular to its own plane		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$	$\frac{3}{2}$
Annular disc inner radius = R_1 and outer radius = R_2	Passing through the centre and perpendicular to the plane		$\frac{M}{2}[R_1^2 + R_2^2]$	–	–
Annular disc	Diameter		$\frac{M}{4}[R_1^2 + R_2^2]$	–	–

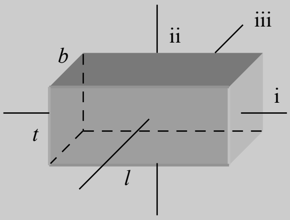
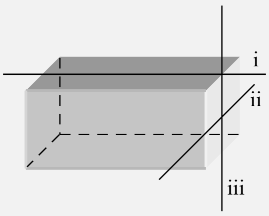
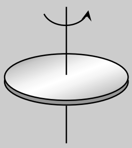
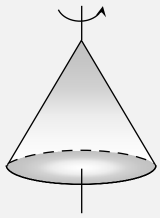
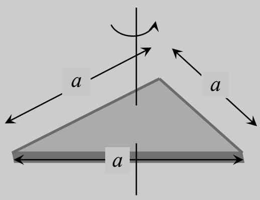
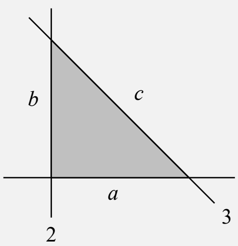
Body	**Axis of Rotation**	**Figure**	**Moment of inertia**	***k***	***k²/R²***
Annular disc	Tangential and Parallel to the diameter		$\frac{M}{4}[5R_1^2 + R_2^2]$	–	–
Annular disc	Tangential and perpendicular to the plane		$\frac{M}{2}[3R_1^2 + R_2^2]$	–	–
Solid cylinder	About its own axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Solid cylinder	Tangential (Generator)		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$	$\frac{3}{2}$
Solid cylinder	About an axis passing through its C.G. and perpendicular to its own axis		$M\left[\frac{L^2}{12} + \frac{R^2}{4}\right]$	$\sqrt{\frac{L^2}{12} + \frac{R^2}{4}}$	
Solid cylinder	About the diameter of one of faces of the cylinder		$M\left[\frac{L^2}{3} + \frac{R^2}{4}\right]$	$\sqrt{\frac{L^2}{3} + \frac{R^2}{4}}$	

Body	**Axis of Rotation**	**Figure**	**Moment of inertia**	***k***	***k*²/*R*²**
Cylindrical shell	About its own axis		MR^2	R	1
Cylindrical shell	Tangential (Generator)		$2MR^2$	$\sqrt{2}R$	2
Cylindrical shell	About an axis passing through its C.G. and perpendicular to its own axis		$M\left[\frac{L^2}{12}+\frac{R^2}{2}\right]$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{2}}$	
Cylindrical shell	About the diameter of one of faces of the cylinder		$M\left[\frac{L^2}{3}+\frac{R^2}{2}\right]$	$\sqrt{\frac{L^2}{3}+\frac{R^2}{2}}$	
Hollow cylinder with inner radius = R_1 and outer radius = R_2	Axis of cylinder		$\frac{M}{2}(R_1^2+R_2^2)$		



Body	Axis of Rotation	Figure	Moment of inertia	k	k^2/R^2
Hollow cylinder with inner radius = R_1 and outer radius = R_2	Tangential		$\frac{M}{2}(R_1^2+3R_2^2)$		
Solid Sphere	About its diametric axis		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$	$\frac{2}{5}$
Solid sphere	About a tangential axis		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$	$\frac{7}{5}$
Spherical shell	About its diametric axis		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$	$\frac{2}{3}$
Spherical shell	About a tangential axis		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$	$\frac{5}{3}$
Hollow sphere of inner radius R_1 and outer radius R_2	About its diametric axis		$\frac{2}{5}M\left[\frac{R_2^5-R_1^5}{R_2^3-R_1^3}\right]$		

Body	**Axis of Rotation**	**Figure**	**Moment of inertia**	***k***	k^2/R^2
Hollow sphere	Tangential		$\frac{2M[R_2^5 - R_1^5]}{5(R_2^3 - R_1^3)} + MR_2^2$		
Long thin rod	About on axis passing through its centre of mass and perpendicular to the rod.		$\frac{ML^2}{12}$	$\frac{L}{\sqrt{12}}$	
Long thin rod	About an axis passing through its edge and perpendicular to the rod		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$	
Rectangular lamina of length *l* and breadth *b*	Passing through the centre of mass and perpendicular to the plane		$\frac{M}{12}[l^2 + b^2]$		
Rectangular lamina	Tangential perpendicular to the plane and at the mid-point of breadth		$\frac{M}{12}[4l^2 + b^2]$		
Rectangular lamina	Tangential perpendicular to the plane and at the mid-point of length		$\frac{M}{12}[l^2 + 4b^2]$		

Body	**Axis of Rotation**	**Figure**	**Moment of inertia**	***k***	***k*²/*R*²**
Rectangular parallelopiped length l, breadth b, thickness t	Passing through centre of mass and parallel to (i) Length (x) (ii) breadth (z) (iii) thickness (y)		(i) $\frac{M[b^2+t^2]}{12}$ (ii) $\frac{M[l^2+t^2]}{12}$ (iii) $\frac{M[b^2+l^2]}{12}$		
Rectangular parallelepiped length l, breath b, thickness t	Tangential and parallel to (i) length (x) (ii) breadth (y) (iii) thickness(z)		(i) $\frac{M}{12}[3l^2+b^2+t^2]$ (ii) $\frac{M}{12}[l^2+3b^2+t^2]$ (iii) $\frac{M}{12}[l^2+b^2+3t^2]$		
Elliptical disc of semimajor axis = a and semiminor axis = b	Passing through CM and perpendicular to the plane		$\frac{M}{4}[a^2+b^2]$		
Solid cone of radius R and height h	Axis joining the vertex and centre of the base		$\frac{3}{10}MR^2$		
Equilateral triangular lamina with side a	Passing through CM and perpendicular to the plane		$\frac{Ma^2}{6}$		
Right angled triangular lamina of sides a, b, c	Along the edges		(1) $\frac{Mb^2}{6}$ (2) $\frac{Ma^2}{6}$ (3) $\frac{M}{6}\left[\frac{a^2b^2}{a^2+b^2}\right]$		