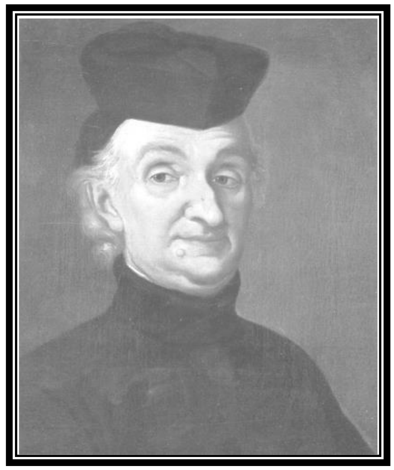
Chapter

6

Hyperbolic Functions

Contents



Vincenzo Riccati

Vincenzo Riccati (1707-1775), who introduced the hyperbolic functions, used Sh. and Ch. for hyperbolic sine and cosine. He used Sc. and Cc. for the circular functions.

Johann Heinrich Lambert (1728-1777) further developed the theory of hyperbolic functions. According to Cajori (vol. 2, page 172), Lambert used sinh and cosh. According to Scott (page 190), Lambert began using sinh and cosh in 1771.

In 1902, George M. Minchin proposed using hysin, hycos, hytan, etc.: "If the prefix hy were put to each of the trigonometric functions, all the names would be pronounceable and not too long." The proposal appeared in Nature, vol. 65 (April 10, 1902).

Hyperbolic Functions

6.1 Definition

We know that parametric co-ordinates of any point on the unit circle $x^2 + y^2 = 1$ is $(\cos\theta, \sin\theta)$; so that these functions are called circular functions and co-ordinates of any point on unit hyperbola $x^2 - y^2 = 1$ is $\left(\frac{e^\theta + e^{-\theta}}{2}, \frac{e^\theta - e^{-\theta}}{2}\right)$ *i.e.,* $(\cosh\theta, \sinh\theta)$. It means that the relation which exists amongst $\cos\theta, \sin\theta$ and unit circle, that relation also exist amongst $\cosh\theta, \sinh\theta$ and unit hyperbola. Because of this reason these functions are called as Hyperbolic functions.

For any (real or complex) variable quantity x,

(1) $\sinh x = \frac{e^x - e^{-x}}{2}$ [Read as 'hyperbolic sine x'] (2) $\cosh x = \frac{e^x + e^{-x}}{2}$ [Read as 'hyperbolic cosine x']

(3) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(4) $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(5) $\text{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

(6) $\text{sec h} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

Note : ❑ $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$

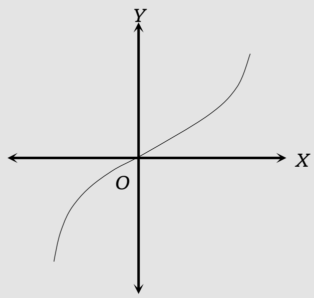
6.2 Domain and Range of Hyperbolic Functions

Let x is any real number

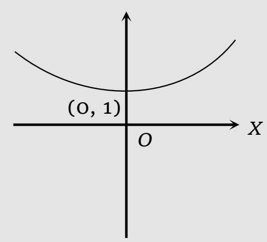
Function	Domain	Range
$\sinh x$	R	R
$\cosh x$	R	$[1, \infty)$
$\tanh x$	R	$(-1, 1)$
$\coth x$	R_0	$R - [-1, 1]$
$\text{sech } x$	R	$(0, 1]$
$\text{cosech } x$	R_0	R_0

6.3 Graph of Real Hyperbolic Functions

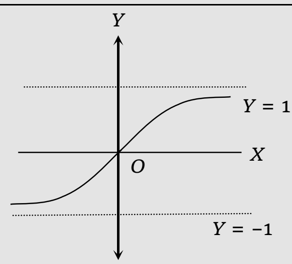
(1) sinh x



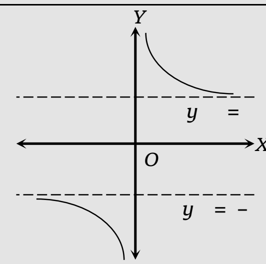
(2) cosh x



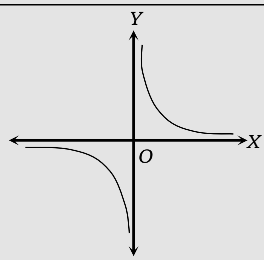
(3) tanh x



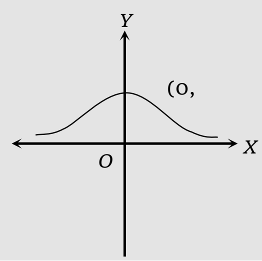
(4) coth x



(5) cosech x



(6) sech x



6.4 Formulae for Hyperbolic Functions

The following formulae can easily be established directly from above definitions

(1) **Reciprocal formulae**

(i) $\operatorname{cosech} x = \dfrac{1}{\sinh x}$ (ii) $\operatorname{sech} x = \dfrac{1}{\cosh x}$ (iii) $\coth x = \dfrac{1}{\tanh x}$

(iv) $\tanh x = \dfrac{\sinh x}{\cosh x}$ (v) $\coth x = \dfrac{\cosh x}{\sinh x}$

(2) **Square formulae**

(i) $\cosh^2 x - \sinh^2 x = 1$ (ii) $\operatorname{sech}^2 x + \tanh^2 x = 1$

(iii) $\coth^2 x - \operatorname{cosech}^2 x = 1$ (iv) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(3) **Expansion or Sum and difference formulae**

(i) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ (ii) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(iii) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(4) **Formulae to transform the product into sum or difference**

(i) $\sinh x + \sinh y = 2\sinh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$ (ii) $\sinh x - \sinh y = 2\cos h\dfrac{x+y}{2}\sin h\dfrac{x-y}{2}$

(iii) $\cosh x + \cosh y = 2\cosh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$ (iv) $\cosh x - \cosh y = 2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$

(v) $2\sinh x\cosh y = \sinh(x+y) + \sinh(x-y)$ (vi) $2\cosh x\sinh y = \sinh(x+y) - \sinh(x-y)$

(vii) $2\cosh x\cosh y = \cosh(x+y) + \cosh(x-y)$ (viii) $2\sinh x\sinh y = \cosh(x+y) - \cosh(x-y)$

(ix) $\cosh x + \sinh x = e^x$ (x) $\cosh x - \sinh x = e^{-x}$ (xi) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

(5) **Trigonometric ratio of multiple of an angle**

(i) $\sinh 2x = 2\sinh x\cosh x = \dfrac{2\tanh x}{1-\tanh^2 x}$

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

(iii) $2\cosh^2 x = \cosh 2x + 1$ (iv) $2\sinh^2 x = \cosh 2x - 1$ (v) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

(vi) $\sinh 3x = 3\sinh x + 4\sinh^3 x$ (vii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$ (viii) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

(6) (i) $\cosh x + \sinh x = e^x$ (ii) $\cosh x - \sinh x = e^{-x}$ (iii)
$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

Example: 1 $\dfrac{\sinh x - \sinh y}{\cosh x - \cosh y}$ is equal to

(a) $2\coth(x+y)$ (b) $\tanh\left(\dfrac{x+y}{2}\right)$ (c) $\coth\left(\dfrac{x+y}{2}\right)$ (d) $\coth\left(\dfrac{x-y}{2}\right)$

Solution: (c) $\dfrac{\sinh x - \sinh y}{\cosh x - \cosh y} = \dfrac{2\cosh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}}{2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}} = \coth\left(\dfrac{x+y}{2}\right)$.

Example: 2 If $\tanh^2 x = \tan^2\theta$, then $\cosh 2x$ is equal to **[EAMCET 1998]**

(a) $-\sin 2\theta$ (b) $\sec 2\theta$ (c) $\cos 3\theta$ (d) $\cos 2\theta$

Solution: (b) $\cosh 2x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x} = \dfrac{1+\tan^2\theta}{1-\tan^2\theta} = \dfrac{1}{\dfrac{1-\tan^2\theta}{1+\tan^2\theta}} = \dfrac{1}{\cos 2\theta} = \sec 2\theta$.

Example: 3 If $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$, then $\cosh u$ is equal to **[EAMCET 1991; Rajasthan PET 1999]**

(a) $\sec x$ (b) $cosec\ x$ (c) $\tan x$ (d) $\sin x$

Solution: (a) $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right) \Rightarrow \therefore \dfrac{e^u}{1} = \dfrac{1+\tan\dfrac{x}{2}}{1-\tan\dfrac{x}{2}}$

$$\cosh u = \frac{e^u + e^{-u}}{2} = \frac{e^{2u}+1}{2e^u} = \frac{\left(\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)^2 + 1}{2.\left(\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)} = \frac{2\left[1+\tan^2\frac{x}{2}\right]}{2\left(1-\tan\frac{x}{2}\right)\left(1+\tan\frac{x}{2}\right)} = \frac{1+\tan^2\frac{\pi}{2}}{1-\tan^2\frac{\pi}{2}} = \frac{1}{\dfrac{1-\tan^2\frac{\pi}{2}}{1+\tan^2\frac{\pi}{2}}} = \frac{1}{\cos x} = \sec x.$$

Example: 4 $\cosh 2 + \sinh 2 =$ **[EAMCET 2000]**

(a) $\frac{1}{e}$ (b) e (c) $\frac{1}{e^2}$ (d) e^2

Solution: (d) $\cosh 2 + \sinh 2 = \frac{e^2+e^{-2}}{2} + \frac{e^2-e^{-2}}{2} = e^2.$

Example: 5 If $\cosh\alpha = \sec x$, then $\tan^2\frac{x}{2} =$ **[Rajasthan PET 1998]**

(a) $\cos^2\frac{\alpha}{2}$ (b) $\sin^2\frac{\alpha}{2}$ (c) $\cot^2\frac{\alpha}{2}$ (d) $\tanh^2\frac{\alpha}{2}$

Solution: (d) $\cosh\alpha = \sec x = \frac{1}{\cos x} = \frac{1}{\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$$\frac{\cosh\alpha}{1} = \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}} \Rightarrow \frac{\cosh\alpha - 1}{\cosh\alpha + 1} = \frac{2\tan^2\frac{x}{2}}{2} = \tan^2\frac{x}{2} \Rightarrow \frac{2\sinh^2\frac{\alpha}{2}}{2\cosh^2\frac{\alpha}{2}} = \tan^2\frac{x}{2} \Rightarrow \tanh^2\frac{\alpha}{2} = \tan^2\frac{x}{2}.$$

6.5 Transformation of a Hyperbolic Functions

Since, $\cosh^2 x - \sinh^2 x = 1$

$\Rightarrow \sinh x = \sqrt{\cosh^2 x - 1}$ $\qquad \Rightarrow \sinh x = \frac{\sqrt{1-\text{sech}^2 x}}{\text{sech}\, x}$

$\Rightarrow \sinh x = \frac{\tanh x}{\sqrt{1-\tanh^2 x}}$ $\qquad \Rightarrow \sinh x = \frac{1}{\sqrt{\coth^2 x - 1}}$

Also, $\sinh x = \frac{1}{\text{cosech}\, x}$

In a similar manner we can express $\cosh x$, $\tanh x, \coth x$,................ in terms of other hyperbolic functions.

6.6 Expansion of Hyperbolic Functions

(1) $\sinh x = \dfrac{e^{x} - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + ...$

(2) $\cosh x = \dfrac{e^{x} + e^{-x}}{2} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + ...$

(3) $\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = x - \dfrac{x^3}{3} + 2x^5 - \dfrac{17}{315}x^7 + ...$

The expansion of $\coth x, \text{cosech } x$ does not exist because $\coth(0) = \infty, \text{cosech}(0) = \infty$.

6.7 Relation between Hyperbolic and Circular Functions

We have from Euler formulae,

$e^{ix} = \cos x + i\sin x$ ....(i) and $e^{-ix} = \cos x - i\sin x$ ....(ii)

Adding (i) and (ii) $\Rightarrow \cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

Subtracting (ii) from (i) $\Rightarrow \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Replacing x by ix in these values, we get $\cos(ix) = \dfrac{e^{-x} + e^{x}}{2} = \cosh x$

$\therefore \cos(ix) = \cosh x$

$\sin(ix) = \dfrac{e^{-x} - e^{x}}{2i} = i\left(\dfrac{e^{x} - e^{-x}}{2}\right)$

$\therefore \sin(ix) = i\sinh x$

Also $\tan(ix) = \dfrac{\sin(ix)}{\cos(ix)} = \dfrac{i\sinh x}{\cosh x}$

$\tan(ix) = i\tanh x$

Similarly replacing x by ix in the definitions of $\sinh x$ and $\cosh x$, we get

$\cosh(ix) = \dfrac{e^{ix} + e^{-ix}}{2} = \cos x$

Also, $\tanh(ix) = \dfrac{\sinh(ix)}{\cosh(ix)} = \dfrac{i\sin x}{\cos x} = i\tan x$

Thus, we obtain the following relations between hyperbolic and trigonometrical functions.

(1) $\sin(ix) = i\sinh x$	(2) $\cos(ix) = \cosh x$
$\sinh(ix) = i\sin x$	$\cosh(ix) = \cos x$
$\sinh x = -i\sin(ix)$	$\cosh x = \cos(ix)$

$\sin x = -i\sin h(ix)$	$\cos x = \cos h(ix)$
(3) $\tan(ix) = i\tanh x$ $\tanh(ix) = i\tan x$ $\tanh x = -i\tan(ix)$ $\tan x = -i\tanh(ix)$	(4) $\cot(ix) = -i\coth x$ $\coth(ix) = -i\cot x$ $\coth x = i\cot(ix)$ $\cot x = i\coth(ix)$
(5) $\sec(ix) = \text{sech}x$ $\sec h(ix) = \sec x$ $\sec hx = \sec(ix)$ $\sec x = \text{sech}(ix)$	(6) $\text{cosec}(ix) = -i\,\text{cosech}\, x$ $\text{cosech}(ix) = i\,\text{cosec}x$ $\text{cosech}\, x = i\,\text{cosec}(ix)$ $\text{cosec}x = i\,\text{cosech}(ix)$

Important Tips

☞ *For obtaining any formula given in (5)th article, use the following substitutions in the corresponding formula for trigonometric functions.*

$\sin x \longrightarrow i\sinh x$ $\quad\cos x \longrightarrow \cosh x$ $\quad\tan x \longrightarrow i\tanh x$

$\sin^2 x \longrightarrow -\sinh^2 x$ $\quad\cos^2 x \longrightarrow \cosh^2 x$ $\quad\tan^2 x \longrightarrow -\tanh^2 x$

For example,

For finding out the formula for $\cosh 2x$ *in terms of* $\tanh x$*, replace* $\tan x$ *by* $i\tanh x$ *and* $\tan^2 x$ *by* $\tan^2 x$ *by* $-\tanh^2 x$ *in the following formula of trigonometric function of* $\cos 2x$ *:*

$\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$ *we get,* $\cosh 2x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

6.8 Period of Hyperbolic Functions

If for any function $f(x), f(x+T) = f(x),$ then $f(x)$ is called the **Periodic function** and least positive value of T is called the **Period** of the function.

$\because\ \sinh x = \sinh(2\pi i + x)$

$\cosh x = \cosh(2\pi i + x)$

and $\tanh x = \tanh(\pi i + x)$

Therefore the period of these functions are respectively $2\pi i, 2\pi i$ and πi. Also period of cosech x, sech x and $\coth x$ are respectively $2\pi i$, $2\pi i$ and πi.

Note : ❑ Remember that if the period of $f(x)$ is T, then period of $f(nx)$ will be $\left(\dfrac{T}{n}\right)$.

❑ Hyperbolic function are neither periodic functions nor their curves are periodic but they show the algebraic properties of periodic functions and having imaginary period.

Example: 6 If $\cos(x+iy)=A+iB$, then A equals **[Rajasthan PET 1994]**

(a) $\cos x\cosh y$ (b) $\sin x\sinh y$ (c) $-\sin x\sinh y$ (d) $\cos x\sinh y$

Solution: (a) $\cos(x+iy)=A+iB$

$\Rightarrow \cos x\cos(iy)-\sin x\sin(iy)=A+iB \Rightarrow \cos x\cosh y-i\sin x\sinh y=A+iB$

$\therefore A=\cos x\cosh y$

Example: 7 If $\cos(u+iv)=x+iy$, then x^2+y^2+1 is equal to **[Rajasthan PET 1999]**

(a) $\cos^2 u+\sinh^2 v$ (b) $\sin^2 u+\cosh^2 v$ (c) $\cos^2 u+\cosh^2 v$ (d) $\sin^2 u+\sinh^2 v$

Solution: (c) $\cos(u+iv)=x+iy$

$\Rightarrow \cos u\cos(iv)-\sin u\sin(iv)=x+iy \Rightarrow \cos u\cosh v-i\sin u\sinh v=x+iy$

$\therefore x=\cos u\cosh v$

$y=-\sin u\sinh v$

$x^2+y^2=\cos^2 u.\cosh^2 v+\sin^2 u.\sinh^2 v$

$=(1-\sin^2 u)\cosh^2 v+\sin^2 v[\cosh^2 v-1] = \cosh^2 v-\sin^2 v$

$\therefore x^2+y^2+1=\cosh^2 v+1-\sin^2 u = \cosh^2 v+\cos^2 u$.

Example: 8 The value of $\sec h(i\pi)$ is **[Rajasthan PET 1999]**

(a) – 1 (b) i (c) 0 (d) 1

Solution: (a) $\sec h(i\pi)=\dfrac{2}{e^{\pi i}+e^{-\pi i}} = \sec\pi=-1$.

Example: 9 $\cos ix+i\sin ix$ equals

(a) e^{ix} (b) e^{-ix} (c) e^{x} (d) e^{-x}

Solution: (d) $\cos ix+i\sin ix$

$=\cosh x+i.i\sinh x = \cosh x-\sinh x=\dfrac{e^x+e^{-x}-e^x+e^{-x}}{2} = e^{-x}$.

Example: 10 $\sin\left(\dfrac{\pi}{6}i\right)$ is equal to

(a) $-\dfrac{i}{2}$ (b) $\dfrac{i}{2}$ (c) $i\dfrac{\sqrt{3}}{2}$ (d) $-i\dfrac{\sqrt{3}}{2}$

Solution: (b) $\because \sinh\left(\dfrac{\pi}{6}i\right)=i.\sin\dfrac{\pi}{6}=i.\dfrac{1}{2}=\dfrac{i}{2}$.

Example: 11 $\sec h(\pi i)+cosech\left(\dfrac{\pi}{2}i\right)$ equals

(a) $1-i$ (b) $-1+i$ (c) $-1-i$ (d) $1+i$

Solution: (c) $\sec h(\pi i)+cosech\left(\dfrac{\pi}{2}i\right)= \sec\pi-i\text{cosec}\dfrac{\pi}{2}=-1-i$.

Example: 12 The period of $\cosh\dfrac{\theta}{3}$ is

(a) $6\pi i$ (b) $2\pi i$ (c) πi (d) $9\pi i$

Solution: (a) Since the period of $\cosh\theta$ is $2\pi i$, so the period of $\cosh\dfrac{\theta}{3}$ is $3.2\pi i=6\pi i$.

Example: 13 The period of $\sinh\left(\dfrac{x}{2}\right)$ is

(a) $2\pi i$ (b) 2π (c) $4\pi i$ (d) 4π

Solution: (c) Since period of $\sinh x$ is $2\pi i$, therefore period of $\sinh\left(\dfrac{x}{2}\right)$ will be $4\pi i$.

6.9 Inverse Hyperbolic Functions

If $\sinh y = x$, then y is called the inverse hyperbolic sine of x and it is written as $y = \sinh^{-1} x$. Similarly $\text{cosech}^{-1} x, \cosh^{-1} x, \tanh^{-1} x$ etc. can be defined.

(1) **Domain and range of Inverse hyperbolic function**

Function	Domain	Range
$\sinh^{-1} x$	R	R
$\cosh^{-1} x$	$[1, \infty)$	R
$\tanh^{-1} x$	$(-1, 1)$	R
$\coth^{-1} x$	$R - [-1, 1]$	R_0
$\text{sech}^{-1} x$	$(0, 1]$	R
$\text{cosech}^{-1} x$	R_0	R_0

(2) **Relation between inverse hyperbolic function and inverse circular function**

Method : Let $\sinh^{-1} x = y$

$\Rightarrow \quad x = \sinh y = -i \sin(iy) \;\Rightarrow\; ix = \sin(iy) \;\Rightarrow\; iy = \sin^{-1}(ix)$

$\Rightarrow y = -i \sin^{-1}(ix) \;\Rightarrow\; \sinh^{-1} x = -i \sin^{-1}(ix)$

Therefore we get the following relations

(i) $\sinh^{-1} x = -i \sin^{-1}(ix)$ (ii) $\cosh^{-1} x = -i \cos^{-1} x$ (iii) $\tanh^{-1} x = -i \tan^{-1}(ix)$

(iv) $\text{sech}^{-1} x = -i \sec^{-1} x$ (v) $\text{cosech}^{-1} x = i \,\text{cosec}^{-1}(ix)$

(3) **To express any one inverse hyperbolic function in terms of the other inverse hyperbolic functions**

To express $\sinh^{-1} x$ in terms of the others

(i) Let $\sinh^{-1} x = y \;\Rightarrow\; x = \sinh y \;\Rightarrow\; \text{cosech}\, y = \frac{1}{x} \;\Rightarrow\; y = \text{cosec}^{-1}\left(\frac{1}{x}\right)$

(ii) $\because \cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$

$\because y = \cosh^{-1} \sqrt{1 + x^2} \;\Rightarrow\; \sinh^{-1} x = \cosh^{-1} \sqrt{1 + x^2}$

(iii) $\because \tanh y = \frac{\sinh y}{\cosh y} = \frac{\sinh y}{\sqrt{1 + \sinh^2 y}} = \frac{x}{\sqrt{1 + x^2}}$

$\therefore y = \tanh^{-1} \frac{x}{\sqrt{1 + x^2}} \;\Rightarrow\; \sinh^{-1} x = \tanh^{-1} \frac{x}{\sqrt{1 + x^2}}$

(iv) $\because \coth y = \frac{\sqrt{1 + \sinh^2 y}}{\sinh y} = \frac{\sqrt{1 + x^2}}{x}$

$\therefore\ y = \coth^{-1}\dfrac{\sqrt{1+x^2}}{x} \Rightarrow \sinh^{-1}x = \coth^{-1}\dfrac{\sqrt{1+x^2}}{x}$

(v) $\because\ \operatorname{sech} y = \dfrac{1}{\cosh y} = \dfrac{1}{\sqrt{1+\sinh^2 y}} = \dfrac{1}{\sqrt{1+x^2}}$

$y = \operatorname{sech}^{-1}\dfrac{1}{\sqrt{1+x^2}} \Rightarrow \sinh^{-1}x = \operatorname{sech}^{-1}\dfrac{1}{\sqrt{1+x^2}}$

(vi) Also, $\sinh^{-1}x = \operatorname{cosech}^{-1}\left(\dfrac{1}{x}\right)$

From the above, it is clear that

$\coth^{-1}x = \tanh^{-1}\left(\dfrac{1}{x}\right)$

$\operatorname{sech}^{-1}x = \cosh^{-1}\left(\dfrac{1}{x}\right)$

$\operatorname{cosech}^{-1} = \sinh^{-1}\left(\dfrac{1}{x}\right)$

Note : ❑ If x is real then all the above six inverse functions are single valued.

(4) **Relation between inverse hyperbolic functions and logarithmic functions**

Method : Let $\sinh^{-1}x = y$

$\Rightarrow x = \sinh y = \dfrac{e^y - e^{-y}}{2} \Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$

But $e^y > 0, \forall y$ and $x < \sqrt{x^2+1}$

$\therefore\ e^y = x + \sqrt{x^2+1} \Rightarrow y = \log(x + \sqrt{x^2+1})$

$\therefore\ \sinh^{-1}x = \log(x + \sqrt{x^2+1})$

By the above method we can obtain the following relations between inverse hyperbolic functions and principal values of logarithmic functions.

(i) $\sinh^{-1}x = \log(x + \sqrt{x^2+1})$ $(-\infty < x < \infty)$ (ii) $\cosh^{-1}x = \log(x + \sqrt{x^2-1})$ $(x \ge 1)$

(iii) $\tanh^{-1}x = \dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right)$ $|x| < 1$ (iv) $\coth^{-1}x = \dfrac{1}{2}\log\left(\dfrac{x+1}{x-1}\right)$ $|x| > 1$

(v) $\operatorname{sech}^{-1}x = \log\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$ $0 < x \le 1$ (vi) $\operatorname{cosech}^{-1}x = \log\left(\dfrac{1+\sqrt{1+x^2}}{x}\right)$ $(x \ne 0)$

Note : ❑ Formulae for values of $\operatorname{cosech}^{-1}x, \operatorname{sech}^{-1}x$ and $\coth^{-1}x$ may be obtained by replacing x by $\dfrac{1}{x}$ in the values of $\sinh^{-1}x, \cosh^{-1}x$ and $\tanh^{-1}x$ respectively.

6.10 Separation of Inverse Trigonometric and Inverse Hyperbolic Functions

If $\sin(\alpha + i\beta) = x + iy$ then $(\alpha + i\beta)$, is called the inverse sine of $(x + iy)$. We can write it as,

$$\sin^{-1}(x + iy) = \alpha + i\beta$$

Here the following results for inverse functions may be easily established.

(1) $\cos^{-1}(x+iy) = \frac{1}{2}\cos^{-1}\left[(x^2+y^2) - \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right] + \frac{i}{2}\cosh^{-1}\left[(x^2+y^2) + \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right]$

(2) $\sin^{-1}(x+iy) = \frac{\pi}{2} - \cos^{-1}(x+iy)$

$= \frac{\pi}{2} - \frac{1}{2}\cos^{-1}\left[(x^2+y^2) - \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right] - \frac{i}{2}\cosh^{-1}\left[(x^2+y^2) + \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right]$

(3) $\tan^{-1}(x+iy) = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right) + \frac{i}{2}\tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right) + \frac{i}{4}\log\left[\frac{x^2+(1+y)^2}{x^2+(1-y)^2}\right]$

(4) $\sin^{-1}(\cos\theta + i\sin\theta) = \cos^{-1}(\sqrt{\sin\theta}) + i\sinh^{-1}(\sqrt{\sin\theta})$ or $\cos^{-1}(\sqrt{\sin\theta}) + i\log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

(5) $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}(\sqrt{\sin\theta}) - i\sinh^{-1}(\sqrt{\sin\theta})$ or $\sin^{-1}(\sqrt{\sin\theta}) - i\log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

(6) $\tan^{-1}(\cos\theta + i\sin\theta) = \frac{\pi}{4} + \frac{i}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$, $(\cos\theta) > 0$

and $\tan^{-1}(\cos\theta + i\sin\theta) = \left(-\frac{\pi}{4}\right) + \frac{1}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$, $(\cos\theta) < 0$

Since each inverse hyperbolic function can be expressed in terms of logarithmic function, therefore for separation into real and imaginary parts of inverse hyperbolic function of complex quantities use the appropriate method.

Note : ❑ Both inverse circular and inverse hyperbolic functions are many valued.

Example: 14 If $x = \log(y + \sqrt{y^2+1})$, then $y =$ **[EAMCET 1995]**

(a) $\tanh x$ (b) $\coth x$ (c) $\sinh x$ (d) $\cosh x$

Solution: (c) $x = \sinh^{-1} y \Rightarrow y = \sinh x$.

Example: 15 If $\cosh^{-1} x = \log(2 + \sqrt{3})$, then $x =$ **[EAMCET 2000]**

(a) 2 (b) 1 (c) 3 (d) 5

Solution: (a) $\cosh^{-1} x = \log(x + \sqrt{x^2-1}) = \log(2+\sqrt{3})$

$\therefore x = 2$.

Example: 16 $\log(3 + 2\sqrt{2}) =$ **[Rajasthan PET 1990]**

(a) $\sinh^{-1} 3$ (b) $\cosh^{-1} 3$ (c) $\tanh^{-1} 3$ (d) $\cosh^{-1} 3$

Solution: (b) $\log(3+2\sqrt{2}) = \log(3+\sqrt{8}) = \log(3+\sqrt{9-1}) = \log(3+\sqrt{3^2-1})$

$\log(3+2\sqrt{2}) = \cosh^{-1} 3$.

Example: 17 $\text{sec h}^{-1}\left(\frac{1}{2}\right)$ is **[Rajasthan PET 1998]**

(a) $\log(\sqrt{3}+\sqrt{2})$ (b) $\log(\sqrt{3}+1)$ (c) $\log(2+\sqrt{3})$ (d) None of these

Solution: (c) $\text{sec h}^{-1}\left(\frac{1}{2}\right)=\cosh^{-1}(2)=\log(2+\sqrt{2^2-2})=\log(2+\sqrt{3})$.

Example: 18 $\sinh^{-1}(2^{3/2})$ is **[EAMCET 2003]**

(a) $\log(2+\sqrt{18})$ (b) $\log(3+\sqrt{8})$ (c) $\log(3-\sqrt{8})$ (d) $\log(\sqrt{8}+\sqrt{27})$

Solution: (b) $\sinh^{-1}(2^{3/2})=\log(2^{3/2}+\sqrt{(2^{3/2})^2+1})=\log(3+\sqrt{8})$.

Example: 19 If $\tan^{-1}(\alpha+i\beta)=x+iy$, then $x=$

(a) $\frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1+\alpha^2+\beta^2}\right)$ (c) $\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$ (d) None of these

Solution: (a) $\tan^{-1}(\alpha+i\beta)=x+iy$

$\tan^{-1}(\alpha-i\beta)=x-iy$

$2x=x+iy+x-iy=\tan^{-1}(\alpha+i\beta)+\tan^{-1}(\alpha-i\beta)$

$\therefore\ x=\frac{1}{2}\tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2}=\frac{1}{2}\tan^{-1}\frac{\alpha+i\beta+\alpha-i\beta}{1-(\alpha+i\beta)(\alpha-i\beta)}=\frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$.

Example: 20 If $-\frac{\pi}{2}<x<\frac{\pi}{2}$, then the value of $\log\sec x$ is

(a) $2\coth^{-1}\left(\text{cosec}^2\frac{x}{2}-1\right)$ (b) $2\coth^{-1}\left(\text{cosec}^2\frac{x}{2}+1\right)$ (c) $2\text{cosech}^{-1}\left(\cot^2\frac{x}{2}-1\right)$ (d) $2\text{cosech}^{-1}\left(\cot^2\frac{x}{2}+1\right)$

Solution: (a) Let $\log\sec x=y$; $\therefore\frac{1}{\cos x}=\frac{e^{y/2}}{e^{-y/2}}$

By componendo and Dividendo rule, $\frac{1+\cos x}{1-\cos x}=\frac{e^{y/2}+e^{-y/2}}{e^{y/2}-e^{-y/2}}\Rightarrow\cot^2\left(\frac{x}{2}\right)=\coth\left(\frac{y}{2}\right)$

$\Rightarrow y=2\coth^{-1}\left(\text{cosec}^2\frac{x}{2}-1\right)$.

Example: 21 The value of $\cosh^{-1}(\sec x)$ is

(a) $\log\left(\frac{1+\sin x}{\cos x}\right)$ (b) $\log\left(\frac{1-\sin x}{\cos x}\right)$ (c) $\log\left(\frac{1+\cos x}{\sin x}\right)$ (d) $\log\left(\frac{1-\cos x}{\sin x}\right)$

Solution: (a) Here $\cosh^{-1}(\sec x)=\log(\sec x+\sqrt{\sec^2 x-1})=\log(\sec x+\tan x)=\log\left(\frac{1+\sin x}{\cos x}\right)$.

Example: 22 $2\sinh^{-1}(\theta)$ is equal to

(a) $\sinh^{-1}(2\theta\sqrt{1+\theta^2})$ (b) $\sinh^{-1}(2\theta\sqrt{1-\theta^2})$ (c) $\sinh^{-1}(\theta\sqrt{1+\theta^2})$ (d) None of these

Solution: (a) We know that, $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

Putting the value of $x=i\theta$

$2\sin^{-1}(i\theta)=\sin^{-1}(2i\theta\sqrt{1-i^2\theta^2})$

$2i\sinh^{-1}(\theta)=\sin^{-1}(2i\theta\sqrt{1+\theta^2})$ or $2i\sinh^{-1}(\theta)=i\sinh^{-1}(2\theta\sqrt{1+\theta^2})$ $(\because \sin^{-1}(ix)=i\sinh^{-1}x)$

$\Rightarrow 2\sinh^{-1}(\theta)=\sinh^{-1}(2\theta\sqrt{1+\theta^2})$.

Example: 23 The value of $\sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ is

(a) $\tanh^{-1} x$ (b) $\coth^{-1} x$ (c) $\sinh^{-1}(2x)$ (d) $\cosh^{-1}(2x)$

Solution: (a) Let $x = \tanh y$, then $\frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\text{sech}\, y} = \sinh y$

$\therefore\ \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sinh^{-1}(\sinh y) \Rightarrow y = \tanh^{-1}(x)$.

Example: 24 If $\cos\alpha \cosh\beta = 1$, then β is equal to

(a) $\log \sec\left(\frac{\alpha}{2}\right)$ (b) $\log \tan\alpha$ (c) $\log(\sec\alpha + \tan\alpha)$ (d) $\log \sin\left(\frac{\alpha}{2}\right)$

Solution: (c) $\cos\alpha.\cosh\beta = 1 \Rightarrow \cosh\beta = \sec\alpha \Rightarrow \beta = \cosh^{-1}(\sec\alpha) = \log(\sec\alpha + \sqrt{\sec^2\alpha - 1}) = \log(\sec\alpha + \tan\alpha)$

Example: 25 $\sinh^{-1}(\sinh^{-1}\theta)$ is equal to

(a) $i\theta$ (b) θ (c) $-i\theta$ (d) $\pi + i\theta$

Solution: (b) $\sinh^{-1}(\sinh^{-1}\theta) = -i\sin(i\sinh^{-1}\theta) = -i\sin[i(-i\sin^{-1}(i\theta)] = -\sin[\sin^{-1}(i\theta)] = -i.i\theta = -i^2\theta = \theta$.

Example: 26 If $\sinh^{-1} x = \text{cosech}^{-1} y$,then the correct statement is

(a) $x = y$ (b) $xy = -1$ (c) $xy = 1$ (d) $x + y = 0$

Solution: (c) Given that, $\sinh^{-1} x = \text{cosech}^{-1} y$ or $\sinh^{-1} x = \sinh^{-1}\left(\frac{1}{y}\right)$ or $x = \sinh\left\{\sinh^{-1}\left(\frac{1}{y}\right)\right\}$ or $x = \frac{1}{y} \Rightarrow xy = 1$.

Example: 27 Find real part of $\tan^{-1}(1+i)$

(a) $-\frac{1}{2}\tan^{-1}(2)$ (b) $\frac{1}{2}\tan^{-1}(2)$ (c) $-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$ (d) 0

Solution: (a) Real part = $\frac{1}{2}\tan^{-1}\frac{2(1)}{1-1-1} = -\frac{1}{2}\tan^{-1}(2)$.

Example: 28 Find real part of $\cosh^{-1}(1)$

(a) – 1 (b) 1 (c) 0 (d) None of these

Solution: (c) We know that $\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right)$

$\therefore\ \cosh^{-1}(1) = \log\left(1 + \sqrt{1^2 - 1}\right) = \log 1 = 0$.

Example: 29 Find imaginary part of $\sin^{-1}\left(\frac{5\sqrt{7} - 9i}{16}\right)$

(a) $\log 2$ (b) $-\log 2$ (c) 0 (d) None of these

Solution: (b) $\left[\sin^{-1}\left(\frac{5\sqrt{7}}{16} - \frac{9i}{16}\right)\right] = -\log\left[\sqrt{\frac{9}{16}} + \sqrt{1 + \frac{9}{16}}\right] = -\log(2)$.

Example: 30 Find real part of $\cos^{-1}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\log\left(\frac{\sqrt{3} - 1}{2}\right)$ (d) None of these

Solution: (b) $\because$ Expression $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} - i\log(\sqrt{\sin\theta} + \sqrt{1 + \sin\theta})$

Where $\theta = \frac{\pi}{6}$

$\therefore \; \cos^{-1}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) = \sin^{-1}\sqrt{\frac{1}{2}} - i\log\left(\sqrt{\frac{1}{2}}+\sqrt{1+\frac{1}{2}}\right) = \frac{\pi}{4} - i\log\left(\frac{1+\sqrt{3}}{2}\right) = \frac{\pi}{4} + i\log\left(\frac{\sqrt{3}-1}{2}\right)$

Real part = $\frac{\pi}{4}$, Imaginary part = $\log\left(\frac{\sqrt{3}-1}{2}\right)$.

Example: 31 Find imaginary part of $\sin^{-1}(\text{cosec}\,\theta)$

(a) $\log\left(\cot\frac{\theta}{2}\right)$ (b) $\frac{\pi}{2}$ (c) $\frac{1}{2}\log\left(\cot\frac{\theta}{2}\right)$ (d) None of these

Solution: (a) Let $\sin^{-1}(\text{cosec}\,\theta) = x + iy$

$\therefore \; \text{cosec}\,\theta = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$

By comparing we get, $\sin x\cosh y = \text{cosec}\,\theta$ ......(i) and $\cos x\sinh y = 0$ ....(ii)

From (ii), $\cos x = 0 \Rightarrow x = \frac{\pi}{2}$

$\therefore$ from (i) $\sin\frac{\pi}{2}.\cosh y = \text{cosec}\,\theta$ or $y = \cosh^{-1}(\text{cosec}\,\theta) = \log[\text{cosec}\,\theta]$

$\Rightarrow y = \log[\text{cosec}\,\theta + \cot\theta] = \log\left(\cot\frac{\theta}{2}\right)$

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$\therefore \sin^{-1}(\text{cosec}\,\theta) = \frac{\pi}{2} + i\log\left(\cot\frac{\theta}{2}\right)$

Real part = $\frac{\pi}{2}$, Imaginary part = $\log\left(\cot\frac{\theta}{2}\right)$.

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Assignment

Formulae and Transformation of Hyperbolic Functions

Basic Level

1. The value of $\cosh 2x$ is **[Rajasthan PET 1985, 86, 88, 90, 2002]**

(a) $\cosh^2 x - \sinh^2 x$ (b) $1 + 2\cosh^2 x$ (c) $1 + 2\sinh^2 x$ (d) None of these

2. $\sinh 3z$ equals **[Rajasthan PET 1990, 92]**

(a) $3\sinh z - 4\sinh^3 z$ (b) $4\sinh^3 z - 3\sinh z$ (c) $3\sinh z + 4\sinh^3 z$ (d) None of these

3. Which of the following statement is true **[Rajasthan PET 1988, 94]**

(a) $\sinh^2 x - \cosh^2 x = 1$ (b) $\sinh^2 x + \cosh^2 x = 1$ (c) $\text{sech}^2 x - \tanh^2 x = 1$ (d) $\coth^2 x - \text{cosech}^2 x = 1$

4. $\tan(x+y)$ equals **[Rajasthan PET 1990, 91, 92]**

(a) $\frac{\tanh x + \tanh y}{1 - \tanh x \tanh y}$ (b) $\frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$ (c) $\frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$ (d) $\frac{\tanh x - \tanh y}{1 + \tanh x \tanh y}$

5. $\sinh^2 x$ equals **[Rajasthan PET 1991]**

(a) $\cosh 2x - 1$ (b) $\cosh^2 x + 1$ (c) $\frac{1}{2}(\cosh 2x - 1)$ (d) $\frac{1}{2}(\cosh 2x + 1)$

6. Which of the following functions is not defined at $x = 0$

(a) $\tanh x$ (b) $\text{cosech } x$ (c) $\sin x$ (d) $\text{sech} x$

7. The value of $(\cosh\theta + \sinh\theta)^n$ is

(a) $e^{n\theta}$ (b) $e^{-n\theta}$ (c) $e^{in\theta}$ (d) $e^{-in\theta}$

8. The value of $\frac{e^{2\theta} - 1}{e^{2\theta} + 1}$ is

(a) $\coth\theta$ (b) $\coth 2\theta$ (c) $\tanh\theta$ (d) $\tanh 2\theta$

9. $\left(\frac{1 + \tanh\theta}{1 - \tanh\theta}\right)^5$ is equal to

(a) $e^{10\theta}$ (b) $e^{5\theta}$ (c) 1 (d) – 1

10. $\frac{1 + \tanh x}{1 - \tanh x}$ is equal to

(a) e^{2x} (b) e^{-2x} (c) i (d) – 1

11. If $\cosh z = \sec\theta$, then $\sinh z$ equals

(a) $\text{cosec}\,\theta$ (b) $\cot\theta$ (c) $\tan\frac{\theta}{2}$ (d) $\tan\theta$

12. If $\text{cosec}\,\theta = \coth x$, then the value of $\tan\theta$ is

(a) $\cosh x$ (b) $\sinh x$ (c) $\tanh x$ (d) $\text{cosech } x$

13. If $\cosh y = \sec x$, then the value of $\tanh^2\left(\frac{y}{2}\right)$ is

(a) $\tan^2\left(\frac{x}{2}\right)$ (b) $\cot^2\frac{x}{2}$ (c) $\sin^2\left(\frac{x}{2}\right)$ (d) $\cos^2\frac{x}{2}$

14. $u = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, then the value of $\tanh\frac{u}{2}$ is **[Rajasthan PET 1997]**

(a) $\cot\frac{x}{2}$ (b) $-\cot\frac{x}{2}$ (c) $-\tan\frac{x}{2}$ (d) $\tan\frac{x}{2}$

15. If $\tan\left(\frac{x}{2}\right)\coth\left(\frac{x}{2}\right) = 1$, then the value of $\cos x \cosh x$ is

(a) 1 (b) – 1 (c) $\cos^2 x$ (d) $\sinh^2 x$

16. $\frac{\text{cosech } x}{\sqrt{\text{cosech}^2 x+1}}$ equals

(a) $\tanh x$ (b) $\coth x$ (c) $\text{sech } x$ (d) $\cosh x$

Advance Level

17. If $f(x)=\cosh x+\sinh x$ and $f(p)=f(x).f(y)$, then the value of p is

(a) xy (b) $x-y$ (c) $x+y$ (d) None of these

18. If $f(x)=\cosh x-\sinh x$, then $f(x_1+x_2+......+x_n)$ is equal to

(a) $f(x_1).f(x_2)........f(x_n)$ (b) $f(x_1)+f(x_2)+......+f(x_n)$ (c) 0 (d) 1

19. If $\sin x \cosh y=\cos\theta$ and $\cos x \sinh y=\sin\theta$, then $\sinh^2 y$ equals

(a) $\sin^2 x$ (b) $\cosh^2 x$ (c) $\cos^2 x$ (d) 1

20. If $\tan\theta=\tanh x \cot y$ and $\tan\phi=\tanh x \tan y$, then $\frac{\sin 2\theta}{\sin 2\phi}$ equals

(a) $\frac{\cosh 2x+\cos 2y}{\cosh 2x-\cos 2y}$ (b) $\frac{\cosh 2x-\cos 2y}{\cosh 2x+\cos 2y}$

(c) $\frac{\cos 2x+\cosh 2y}{\cos 2x-\cosh 2y}$ (d) None of these

Relation between Hyperbolic and Circular Functions

Basic Level

21. The value of $\text{cosech}\left(\frac{\pi i}{6}\right)$ is

(a) – 2 (b) 2 (c) – 2 i (d) 2 i

22. $\sinh\left(x+\frac{\pi i}{2}\right)$ equals

(a) $i\cosh x$ (b) $-i\cosh x$ (c) $\cos x$ (d) $-\cos x$

23. $\tanh\left(\frac{\pi i}{4}\right)-\coth\left(\frac{\pi i}{4}\right)$ is equal to

(a) 0 (b) 2 (c) $\sqrt{2}$ (d) None of these

24. $\cos(i^5 x)$ equals **[Rajasthan PET 1989]**

(a) $i\cosh x$ (b) $-i\cosh x$ (c) $\cosh x$ (d) $-\cosh x$

25. If $\sin(x+iy)=A+iB,$ then A equals

(a) $\sinh x\cos y$ (b) $\sin x\cosh y$ (c) $\cos x\sinh y$ (d) $\cosh x\sin y$

26. The imaginary part of $\sin^2(x+iy)$ is

(a) $\frac{1}{2}\cosh 2x\cos 2y$ (b) $\frac{1}{2}\cos 2x\cosh 2y$ (c) $\frac{1}{2}\sinh 2x\sin 2y$ (d) $\frac{1}{2}\sin 2x\sinh 2y$

27. Imaginary part of $\cosh(\alpha+i\beta)-\cosh(\alpha-i\beta)=$ **[Rajasthan PET 2000]**

(a) $2\sinh\alpha\sinh\beta$ (b) $2\sinh\alpha\sin\beta$ (c) $\cosh\alpha\cos\beta$ (d) $2\cos\alpha\cosh\beta$

28. Real part of $\cosh(\alpha+i\beta)$ is **[Rajasthan PET 1995]**

(a) $\cosh\alpha\cos\beta$ (b) $\cos\alpha\cos\beta$ (c) $\cos\alpha\cosh\beta$ (d) $\sin\alpha\sinh\beta$

29. The value of $\sinh(x+2\pi ni)$ is

(a) $\frac{e^x+e^{-x}}{2}$ (b) $\frac{e^x-e^{-x}}{2}$ (c) $\frac{e^x-e^{-x}}{2i}$ (d) $\frac{e^x+e^{-x}}{2i}$

30. $\sin^2(ix)+\cosh^2 x$ is equal to

(a) 1 (b) – 1 (c) $2\cosh^2 x$ (d) $\cosh 2x$

31. The period of $\coth\left(\frac{nx}{4}\right)$ is

(a) $\frac{\pi i}{n}$ (b) $\frac{4\pi i}{n}$ (c) $\frac{n\pi i}{4}$ (d) πi

32. The period of e^z is

(a) 2π (b) π (c) $2\pi i$ (d) πi

33. The period of $\cosh(4x)$ is

(a) $2\pi i$ (b) πi (c) $\frac{\pi i}{2}$ (d) 2π

Advance Level

34. If $\tan(\theta+i\phi)=\sin(x+iy)$, then the value of $\coth y\sinh 2\phi$ is

(a) $\tan x.\cot 2\theta$ (b) $\tan x\sin 2\theta$ (c) $\cot x\sin 2\theta$ (d) None of these

35. If $\cos(\alpha+i\beta)=\rho(\cos\psi+i\sin\psi)$,then value of $\tan\psi$ is

(a) $\tanh\alpha\tan\beta$ (b) $\frac{-\tanh\alpha\tan\beta}{\rho}$ (c) $-\cot\alpha\coth\beta$ (d) $-\tan\alpha\tanh\beta$

Inverse Hyperbolic Functions

Basic Level

36. $\sinh^{-1}x=$ **[Rajasthan PET 1987, 93, 96, 2000]**

(a) $\log(x+\sqrt{1-x^2})$ (b) $\log(x+\sqrt{x^2+1})$ (c) $\log(x+\sqrt{x^2-1})$ (d) None of these

37. $\cosh^{-1} x =$ **[Rajasthan PET 1988, 90, 92, 2002]**

(a) $\log(x+\sqrt{x^2+1})$ (b) $\log(x-\sqrt{x^2+1})$ (c) $\log(x-\sqrt{x^2-1})$ (d) $\log(x+\sqrt{x^2-1})$

38. $\tanh^{-1} x =$ **[Rajasthan PET 1988, 91, 92, 99]**

(a) $\frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$ (b) $\frac{1}{2}\log\left(\frac{x-1}{x+1}\right)$ (c) $\frac{1}{2}\log\left(\frac{1-x}{1+x}\right)$ (d) $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

39. The value of $\sinh^{-1}(1)$ is **[Rajasthan PET 1989]**

(a) 0 (b) $\log(\sqrt{2}+1)$ (c) $\log(1-\sqrt{2})$ (d) None of these

40. $\coth^{-1} x$ equals **[Rajasthan PET 1990]**

(a) $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ (b) $\frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$ (c) $\frac{1}{2}\log\left(\frac{x-1}{x+1}\right)$ (d) None of these

41. $\text{cosech}^{-1} x$ equals **[Rajasthan PET 1991]**

(a) $\log\left(\frac{1+\sqrt{1+x^2}}{x}\right)$ (b) $\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ (c) $\log\left(\frac{1-\sqrt{1-x^2}}{x}\right)$ (d) $\log\left(\frac{1-\sqrt{1+x^2}}{x}\right)$

42. The value of $2\coth^{-1}\left(\frac{Z}{2}\right)$ is

(a) $\log\left(\frac{Z-2}{Z+2}\right)$ (b) $\frac{1}{2}\log\left(\frac{Z-1}{Z+1}\right)$ (c) $\frac{1}{2}\log\left(\frac{Z+1}{Z-1}\right)$ (d) $-\log\left(\frac{Z-2}{Z+2}\right)$

43. $\text{sech}^{-1}(\sin x)$ equals

(a) $\log\cot\frac{x}{2}$ (b) $\log\tan\frac{x}{2}$ (c) $\log\cot x$ (d) None of these

44. If $\text{cosech}^{-1}(1) = x+iy,$ then the value of y is **[Rajasthan PET 1986]**

(a) 1 (b) 0 (c) $\log(1+\sqrt{2})$ (d) – 1

45. The value of $\tanh^{-1}(2^{-1})$ is

(a) log 2 (b) log 2^{-1}

(c) $\log\sqrt{3}$ (d) None of these

46. If $\log(2+\sqrt{3}) = \cosh^{-1} K$ then K equals

(a) 1 (b) 0 (c) 2 (d) None of these

47. $-i\tan^{-1}(ix)$ equals

(a) $\tanh^{-1} x$ (b) $-\tanh^{-1} x$ (c) $\tanh^{-1}(ix)$ (d) None of these

48. $2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is equal to

(a) $\cosh^{-1}\left(\frac{1}{3}\right)$ (b) $\cosh^{-1}(\sqrt{3})$ (c) $\cosh^{-1}(3)$ (d) $\cosh^{-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$

49. If $\tanh x = \frac{3}{4}$, then the value of x is

(a) $\sqrt{7}$ (b) $-\sqrt{7}$ (c) $\log\sqrt{7}$ (d) $-\log\sqrt{7}$

50. $\tanh^{-1}\left(\frac{1}{2}\right)+\tanh^{-1}\left(\frac{1}{3}\right)$ is equal to

(a) $\tanh^{-1}\left(\frac{5}{7}\right)$ (b) $\tanh^{-1}\left(\frac{7}{5}\right)$ (c) $\tanh^{-1}\left(\frac{1}{6}\right)$ (d) $\tanh^{-1}\left(\frac{5}{6}\right)$

51. $\sinh^{-1}\left(\frac{1}{2}\right)$ is equal to

(a) $\tanh^{-1}(\sqrt{5})$ (b) $\tanh^{-1}\left(\frac{1}{\sqrt{5}}\right)$ (c) $\tanh^{-1}(\sqrt{3})$ (d) $\tanh^{-1}\left(\frac{2}{\sqrt{5}}\right)$

52. $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ is equal to

(a) $\tanh^{-1}\left(\tan\frac{\theta}{2}\right)$ (b) $\tanh^{-1}\left(\tanh\frac{\theta}{2}\right)$ (c) $2\tanh^{-1}\left(\tan\frac{\theta}{2}\right)$ (d) $2\tan^{-1}\left(\tanh\frac{\theta}{2}\right)$

53. If $\sin^{-1}(A+iB)=x+iy$, then $\frac{A}{B}$ equals **[Rajasthan PET 1987]**

(a) $\frac{\tan x}{\tanh y}$ (b) $\frac{\tanh x}{\tan y}$

(c) $\frac{\tanh x}{\tanh y}$ (d) $\frac{\cos x}{\cosh y}$

Advance Level

54. The general value of $\cosh^{-1} x$ is

(a) $2\pi ri+\log(x+\sqrt{x^2+1})$ (b) $2\pi ri+\log(x+\sqrt{x^2-1})$ (c) $\pi ri+(-1)^r\log(x+\sqrt{x^2+1})$ (d) $2\pi ri+(-1)^r\log(x+\sqrt{x^2-1})$

55. If $x=\log\left(\frac{1}{y}+\sqrt{\frac{1}{y^2}+1}\right)$, then y is equal to

(a) $\tanh x$ (b) $\cosh x$ (c) $\sinh x$ (d) $\text{cosech}\, x$

56. The imaginary part of $\tan^{-1}(\cos\theta+i\sin\theta)$ is

(a) $\tanh^{-1}(\sin\theta)$ (b) $\tanh^{-1}(\infty)$ (c) $\frac{1}{2}\tanh^{-1}(\sin\theta)$ (d) None of these

57. If $\cosh^{-1}(p+iq)=u+iv,$ then the equation with roots $\cos^2 u$ and $\cosh^2 v$

(a) $x^2-x(p^2+q^2)+p^2=0$ (b) $x^2-x(p^2+q^2+1)+1=0$ (c) $x^2+x(p^2+q^2+1)+1=0$ (d) $x^2-x(p^2+q^2+1)+p^2=0$

58. The value of $\log\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)$ is

(a) $i\tan^{-1}(\sinh x)$ (b) $-i\tan^{-1}(\sinh x)$ (c) $i\tan^{-1}(\cosh x)$ (d) None of these

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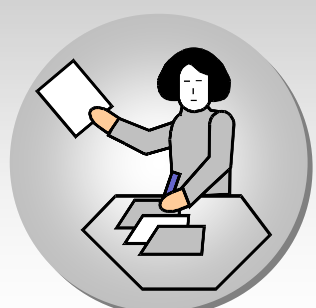

Answer Sheet

Hyperbolic Functions ***Assignment (Basic & Advance Level)***

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
c	c	d	b	c	b	a	c	a	a	d	b	a	d	a	c	c	a	c	a
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
c	a	b	c	b	d	b	a	b	a	b	c	c	c	d	b	d	d	b	b
41	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**		
a	d	a	b	c	c	a	c	c	a	b	c	a	b	d	c	d	a		