

# Determinants and Matrices

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### Assignment (Basic and Advance Level)

### Answer Sheet of Assignment



G. Cramer

*The theory of determinants may be said to have begun with G.W. Leibnitz (1646-1716) who gave a rule for the solution of simultaneous linear equations equivalent to that of the Chinese. This rule was simplified by G. Cramer (1704-1752), a Swiss mathematician, in 1750. E. Bezout (1730-1783), a French mathematician simplified it further in 1764. However, A.T. Vandermonde (1735-1796) gave the first systematic account of determinants in 1771. P.S. Laplace (1749-1827), in 1772, gave the general method of expanding a determinant in terms of minors. In 1773, J.L., Lagrange (1736-1813) treated determinants fo order 2 and 3 and used them for purposes other than the solution of equations. Carl F. Gauss (1777-1855) used determinants in the theory of numbers.*

*J.J. Sylvester, who was very fond of assigning imaginative names to his creations and inventions, wrote down in 1850, certain terms and expressions in the form of a rectangular arrangement, Sylvester gave this rectangular arrangement the name 'matrix'. In particular, the names of Sir William Rowan Hamilton, (1805-1865) and Arthur Cayley deserve special mention. Sir Hamilton, in 1853, and Arthur Cayley, in 1858, made significant contributions to the theory of matrices.*

## 8.1 Determinants

### 8.1.1 Definition

(1) Consider two equations,  $a_1x + b_1y = 0$  .....(i) and  $a_2x + b_2y = 0$  .....(ii)

Multiplying (i) by  $b_2$  and (ii) by  $b_1$  and subtracting, dividing by  $x$ , we get,  $a_1b_2 - a_2b_1 = 0$

The result  $a_1b_2 - a_2b_1$  is represented by  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Which is known as determinant of order two and  $a_1b_2 - a_2b_1$  is the expansion of this determinant. The horizontal lines are called rows and vertical lines are called columns.

Now let us consider three homogeneous linear equations

$$a_1x + b_1y + c_1z = 0, \quad a_2x + b_2y + c_2z = 0 \quad \text{and} \quad a_3x + b_3y + c_3z = 0$$

Eliminated  $x, y, z$  from above three equations we obtain

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0 \quad \text{.....(iii)}$$

The L.H.S. of (iii) is represented by  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Its contains three rows and three columns, it is called a determinant of third order.

**Note :**  $\square$  The number of elements in a second order is  $2^2 = 4$  and the number of elements in a third order determinant is  $3^2 = 9$ .

(2) **Rows and columns of a determinant :** In a determinant horizontal lines counting from top 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>,..... respectively known as rows and denoted by  $R_1, R_2, R_3, \dots$  and vertical lines counting left to right, 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>,..... respectively known as columns and denoted by  $C_1, C_2, C_3, \dots$

(3) **Shape and constituents of a determinant :** Shape of every determinant is square. If a determinant of  $n$  order then it contains  $n$  rows and  $n$  columns.

*i.e.*, Number of constituents in determinants =  $n^2$

(4) **Sign system for expansion of determinant :** Sign system for order 2, order 3, order 4,.....

are given by  $\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}, \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}, \dots$

### 8.1.2 Expansion of Determinants

Unlike a matrix, determinant is not just a table of numerical data but (quite differently) a short hand way of writing algebraic expression, whose value can be computed when the values of terms or elements are known.

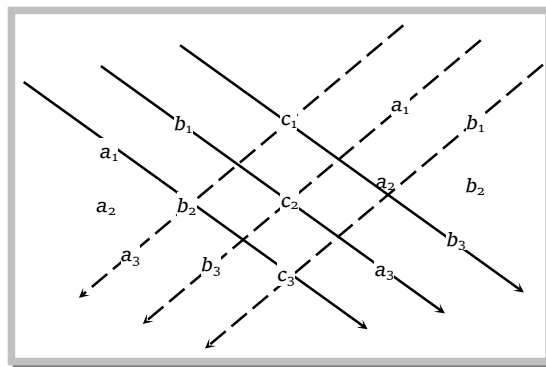
(1) The 4 numbers  $a_1, b_1, a_2, b_2$  arranged as  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$  is a determinant of second order. These numbers are called elements of the determinant. The value of the determinant is defined as  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$ .

The expanded form of determinant has  $2!$  terms.

(2) The 9 numbers  $a_r, b_r, c_r (r = 1, 2, 3)$  arranged as  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  is a determinant of third order. Take any row (or column); the value of the determinant is the sum of products of the elements of the row (or column) and the corresponding determinant obtained by omitting the row and the column of the element with a proper sign, given by the rule  $(-1)^{i+j}$ , where  $i$  and  $j$  are the number of rows and the number of columns respectively of the element of the row (or the column) chosen. Thus  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

The diagonal through the left-hand top corner which contains the element  $a_1, b_2, c_3$  is called the leading diagonal or principal diagonal and the terms are called the leading terms. The expanded form of determinant has  $3!$  terms.

**Short cut method or Sarrus diagram method :** To find the value of third order determinant, following method is also useful



Taking product of R.H.S. diagonal elements positive and L.H.S. diagonal elements negative and adding them. We get the value of determinant as  $= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3$

**Note :** □ This method does not work for determinants of order greater than three.

**8.1.3 Evaluation of Determinants**

If  $A$  is a square matrix of order 2, then its determinant can be easily found. But to evaluate determinants of square matrices of higher orders, we should always try to introduce zeros at maximum number of places in a particular row (column) by using the properties and then we should expand the determinant along that row (column).

We shall be using the following notations to evaluate a determinant :

(1)  $R_i$  to denote  $i^{\text{th}}$  row.

(2)  $R_i \leftrightarrow R_j$  to denote the interchange of  $i^{\text{th}}$  and  $j^{\text{th}}$  rows.

(3)  $R_i \rightarrow R_i + \lambda R_j$  to denote the addition of  $\lambda$  times the elements of  $j^{\text{th}}$  row to the corresponding elements of  $i^{\text{th}}$  row.

(4)  $R_i(\lambda)$  to denote the multiplication of all element of  $i^{\text{th}}$  row by  $\lambda$ .

Similar notations are used to denote column operations if  $R$  is replaced by  $C$ .

### 8.1.4 Properties of Determinants

**P-1 :** The value of determinant remains unchanged, if the rows and the columns are interchanged.

If  $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  and  $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ . Then  $D' = D$ ,  $D$  and  $D'$  are transpose of each other.

**Note :**  $\square$  Since the determinant remains unchanged when rows and columns are interchanged, it is obvious that any theorem which is true for 'rows' must also be true for 'columns'.

**P-2 :** If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value but is changed in sign only.

Let  $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  and  $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ . Then  $D' = -D$

**P-3 :** If a determinant has two rows (or columns) identical, then its value is zero.

Let  $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ . Then,  $D = 0$

**P-4 :** If all the elements of any row (or column) be multiplied by the same number, then the value of determinant is multiplied by that number.

Let  $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  and  $D' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ . Then  $D' = kD$

**P-5 :** If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the determinants.

$$\text{e.g., } \begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

**P-6 :** The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column)

$$\text{e.g., } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3-na_1 & b_3-nb_1 & c_3-nc_1 \end{vmatrix}. \text{ Then } D' = D$$

**Note :** □ It should be noted that while applying **P-6** at least one row (or column) must remain unchanged.

**P-7 :** If all elements below leading diagonal or above leading diagonal or except leading diagonal elements are zero then the value of the determinant equal to multiplied of all leading diagonal elements.

$$\text{e.g., } \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

**P-8 :** If a determinant  $D$  becomes zero on putting  $x = \alpha$ , then we say that  $(x - \alpha)$  is factor of determinant.

$$\text{e.g., if } D = \begin{vmatrix} x & 5 & 2 \\ x^2 & 9 & 4 \\ x^3 & 16 & 8 \end{vmatrix}. \text{ At } x = 2, D = 0 \text{ (because } C_1 \text{ and } C_2 \text{ are identical at } x = 2)$$

Hence  $(x - 2)$  is a factor of  $D$ .

**Note :** □ It should be noted that while applying operations on determinants then at least one row (or column) must remain unchanged. or, Maximum number of operations = order of determinant - 1

□ It should be noted that if the row (or column) which is changed by multiplied a non zero number, then the determinant will be divided by that number.

**Example: 1** If  $n \neq 3k$  and  $1, \omega, \omega^2$  are the cube roots of unity, then  $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$  has the value

[Pb. CET 1991; Rajasthan PET 2001; AIEEE 2003]

- (a) 0 (b)  $\omega$  (c)  $\omega^2$  (d) 1

**Solution:** (a) Applying  $C_1 \rightarrow C_1 + C_2 + C_3$ , we get

$$\Delta = \begin{vmatrix} 1+\omega^n+\omega^{2n} & \omega^n & \omega^{2n} \\ 1+\omega^n+\omega^{2n} & 1 & \omega^n \\ 1+\omega^n+\omega^{2n} & \omega^{2n} & 1 \end{vmatrix} = \begin{vmatrix} 0 & \omega^n & \omega^{2n} \\ 0 & 1 & \omega^n \\ 0 & \omega^{2n} & 1 \end{vmatrix} = 0 \quad (\because 1 + \omega^n + \omega^{2n} = 0 \text{ if } n \text{ is not multiple of } 3)$$

**Example: 2**  $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} =$

[Rajasthan PET 1992; Kerala (Engg.) 2002]

- (a)  $xyz \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$  (b)  $xyz$  (c)  $1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$  (d)  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

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**Solution: (a)**  $\Delta = xyz \begin{vmatrix} 1 + \frac{1}{x} & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & 1 + \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & 1 + \frac{1}{z} \end{vmatrix} = xyz \left( 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \begin{vmatrix} \frac{1}{x} & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & 1 + \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & 1 + \frac{1}{z} \end{vmatrix}$  (by  $R_1 \rightarrow R_1 + R_2 + R_3$ )

$= xyz \left( 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \begin{vmatrix} \frac{1}{x} & 0 & 0 \\ \frac{1}{y} & 1 & 0 \\ \frac{1}{z} & 0 & 1 \end{vmatrix}$  (by  $C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$ )

$= xyz \left( 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \begin{vmatrix} \frac{1}{x} & 0 \\ 0 & 1 \end{vmatrix} = xyz \left( 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$

**Trick :** Put  $x = 1, y = 2$  and  $z = 3$ , then  $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2(11) - 1(3) + 1(1 - 3) = 17$ .

option (a) gives  $1 \times 2 \times 3 \left( 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right) = 17$

**Example: 3** The value of  $\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix}$  is equal to zero, where  $m$  is

- (a) 6 (b) 4 (c) 5 (d) None of these

**Solution: (c)**  $\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0$

Applying  $C_2 \rightarrow C_1 + C_2$

$\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_4 + {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_6 + {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_8 + {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0 \Rightarrow \Delta = \begin{vmatrix} {}^{10}C_4 & {}^{11}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{12}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{13}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0$

Clearly  $m = 5$  satisfies the above result [ $\because C_2, C_3$  will be identical]

**Example: 4** If  $a_1, a_2, a_3, \dots, a_n, \dots$  are in G.P. then the value of the determinant  $\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$  is

[AIEEE 2004; IIT 1993]

- (a) -2 (b) 1 (c) 2 (d) 0

**Solution: (d)**  $\because a_1, a_2, a_3, \dots, a_n, \dots$  are in G.P.

$\therefore a_{n+1}^2 = a_n \cdot a_{n+2} \Rightarrow 2 \log a_{n+1} = \log a_n + \log a_{n+2}$

$a_{n+4}^2 = a_{n+3} \cdot a_{n+5} \Rightarrow 2 \log a_{n+4} = \log a_{n+3} + \log a_{n+5}$

$a_{n+7}^2 = a_{n+6} \cdot a_{n+8} \Rightarrow 2 \log a_{n+7} = \log a_{n+6} + \log a_{n+8}$

Putting these values in the second column of the given determinant, we get

$\Delta = \frac{1}{2} \begin{vmatrix} \log a_n & \log a_n + \log a_{n+2} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+3} + \log a_{n+5} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+6} + \log a_{n+8} & \log a_{n+8} \end{vmatrix}$

$$= \frac{1}{2}(0) = 0 \quad [ \because C_2 \text{ is the sum of two elements, first identical with } C_1 \text{ and second with } C_3 ]$$

**Example: 5** The value of  $\begin{vmatrix} 1 & 1 & 1 \\ (2^x + 2^{-x})^2 & (3^x + 3^{-x})^2 & (5^x + 5^{-x})^2 \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix}$

- (a) 0 (b)  $30^x$  (c)  $30^{-x}$  (d) None of these

**Solution:** (a) Applying  $R_2 \rightarrow R_2 - R_3$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 \cdot 2^x \cdot 2 \cdot 2^{-x} & 2 \cdot 3^x \cdot 2 \cdot 3^{-x} & 2 \cdot 5^x \cdot 2 \cdot 5^{-x} \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix} = 4 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix} = 0 \quad [ \because R_1 \text{ and } R_2 \text{ are identical } ]$$

**Trick :** Putting  $x = 0$ , we get option (a) is correct

**Example: 6** If  $x, y, z$  are integers in A.P. lying between 1 and 9 and  $x51, y41$  and  $z31$  are three digit numbers then

the value of  $\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$  is

- (a)  $x + y + z$  (b)  $x - y + z$  (c) 0 (d) None of these

**Solution:** (c)  $\because x51 = 100x + 50 + 1,$   
 $y41 = 100y + 40 + 1$   
 $z31 = 100z + 30 + 1$

$$\therefore \Delta = \begin{vmatrix} 5 & 4 & 3 \\ 100x + 50 + 1 & 100y + 40 + 1 & 100z + 30 + 1 \\ x & y & z \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - 100R_3 - 10R_1$

$$\Delta = \begin{vmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = x - 2y + z$$

$\because x, y, z$  are in A.P.,  $\therefore x - 2y + z = 0, \therefore \Delta = 0$

**Example: 7** If  $a \neq b \neq c$ , the value of  $x$  which satisfies the equation  $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$  is

[EAMCET 1988; Karnataka CET 1991; MNR 1980; MP PET 1988, 99, 2001; DCE 2001]

- (a)  $x = 0$  (b)  $x = a$  (c)  $x = b$  (d)  $x = c$

**Solution:** (a) Expanding determinant, we get,  $\Delta = -(x-a)[-(x+b)(x-c)] + (x+b)[(x+a)(x+c)] = 0 \Rightarrow 2x^3 - (2\Sigma ab)x = 0$   
 $\Rightarrow$  Either  $x = 0$  or  $x^2 = \Sigma ab$ . Since  $x = 0$  satisfies the given equation.

**Trick :** On putting  $x = 0$ , we observe that the determinant becomes  $\Delta_{x=0} = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

$\therefore x = 0$  is a root of the given equation.

**Example: 8** The number of distinct real roots of  $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$  in the interval  $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$  is

- (a) 0 (b) 2 (c) 1 (d) 3

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**Solution: (c)**  $(2 \cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$

Applying,  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$

$$(2 \cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = 0 \Rightarrow (2 \cos x + \sin x)(\sin x - \cos x)^2 = 0$$

$\therefore \tan x = -2, 1$  But  $\tan x \neq -2$  in  $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ . Hence  $\tan x = 1 \Rightarrow x = \frac{\pi}{4}$

**Example: 9** If  $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda + 1 & 2 - \lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$ , then value of  $t$  is [IIT 1981]

- (a) 16 (b) 18 (c) 17 (d) 19

**Solution: (b)** Since it is an identity in  $\lambda$  so satisfied by every value of  $\lambda$ . Now put  $\lambda = 0$  in the given equation, we have

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & -4 \\ -3 & 4 & 0 \end{vmatrix} = -12 + 30 = 18$$

**Example: 10** If  $f(x) = \begin{vmatrix} 1 & x & (x+1) \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$ , then  $f(100)$  is equal to [IIT 1999, MP PET 2000]

- (a) 0 (b) 1 (c) 100 (d) -100

**Solution: (a)**  $f(x) = \begin{vmatrix} 1 & x & (x+1) \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$

Applying  $C_3 \rightarrow C_3 - C_2$ , we get

$$f(x) = \begin{vmatrix} 1 & x & 1 \\ 2x & x(x-1) & 2x \\ 3x(x-1) & x(x-1)(x-2) & 3x(x-1) \end{vmatrix} = 0 \text{ . Hence } f(100) = 0$$

**Example: 11** The value of  $\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 4 & 4 & 3 & 0 & 0 \\ 5 & 5 & 5 & 4 & 0 \\ 6 & 6 & 6 & 6 & 5 \end{vmatrix}$  is

- (a) 6! (b) 5! (c)  $1.2^2.3.4^3.5^4.6^4$  (d) None of these

**Solution: (b)** The elements in the leading diagonal are 1, 2, 3, 4, 5. On one side of the leading diagonal all the elements are zero.

$\therefore$  The value of the determinant

$$= \text{The product of the elements in the leading diagonal} = 1.2.3.4.5 = 5!$$

**Example: 12** The determinant  $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$  if [DCE 2000, 2001]

- (a)  $a, b, c$  are in A.P.  
 (b)  $a, b, c$  are in G.P. or  $(x - \alpha)$  is a factor of  $ax^2 + 2bx + c = 0$   
 (c)  $a, b, c$  are in H.P.



(d)  $\alpha$  is a root of the equation

**Solution:** (b) Applying  $R_3 \rightarrow R_3 - \alpha R_1 - R_2$ , we get

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ 0 & 0 & -a\alpha^2 - b\alpha - b\alpha - c \end{vmatrix} = 0$$

$\Rightarrow -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0 \Rightarrow a\alpha^2 + 2b\alpha + c = 0$  or  $b^2 = ac$   
 $\Rightarrow x = \alpha$  is a root of  $ax^2 + 2bx + c = 0$  or  $a, b, c$  are in G.P.  
 $\Rightarrow (x - \alpha)$  is a factor of  $ax^2 + 2bx + c = 0$  or  $a, b, c$  are in G.P.

### 8.1.5 Minors and Cofactors

(1) **Minor of an element** : If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by  $M_{ij}$

Consider the determinant  $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ , then determinant of minors

$$M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix},$$

where  $M_{11} = \text{minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ ,  $M_{12} = \text{minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$   
 $M_{13} = \text{minor of } a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly, we can find the minors of other elements . Using this concept the value of determinant can be

$$\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

or,  $\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$  or,  $\Delta = a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}$ .

(2) **Cofactor of an element** : The cofactor of an element  $a_{ij}$  (i.e. the element in the  $i^{th}$  row and  $j^{th}$  column) is defined as  $(-1)^{i+j}$  times the minor of that element. It is denoted by  $C_{ij}$  or  $A_{ij}$  or  $F_{ij}$ .

$$C_{ij} = (-1)^{i+j} M_{ij}$$

If  $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ , then determinant of cofactors is  $C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$ , where

$$C_{11} = (-1)^{1+1} M_{11} = +M_{11}, C_{12} = (-1)^{1+2} M_{12} = -M_{12} \text{ and } C_{13} = (-1)^{1+3} M_{13} = +M_{13}$$

Similarly, we can find the cofactors of other elements.

**Note** : □ The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant i.e.  $\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$   
 $= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

where the capital letters  $C_{11}, C_{12}, C_{13}$  etc. denote the cofactors of  $a_{11}, a_{12}, a_{13}$  etc.

□ In general, it should be noted

$$a_{i1}C_{j1} + a_{i2}C_{j2} + a_{i3}C_{j3} = 0, \text{ if } i \neq j \text{ or } a_{1i}C_{1j} + a_{2i}C_{2j} + a_{3i}C_{3j} = 0, \text{ if } i \neq j$$

□ If  $\Delta'$  is the determinant formed by replacing the elements of a determinant  $\Delta$  by their corresponding cofactors, then if  $\Delta = 0$ , then  $\Delta^C = 0$ ,  $\Delta' = \Delta^{n-1}$ , where  $n$  is the order of the determinant.

**Example: 13** The cofactor of the element 4 in the determinant  $\begin{vmatrix} 1 & 3 & 5 & 1 \\ 2 & 3 & 4 & 2 \\ 8 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{vmatrix}$  is [MP PET 1987]

(a) 4 (b) 10 (c) -10 (d) -4

**Solution:** (b) The cofactor of element 4, in the 2<sup>nd</sup> row and 3<sup>rd</sup> column is  $(-1)^{2+3} \begin{vmatrix} 1 & 3 & 1 \\ 8 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -\{1(-2) - 3(8 - 0) + 1.16\} = 10$

**Example: 14** If  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  and  $A_1, B_1, C_1$  denote the cofactors of  $a_1, b_1, c_1$  respectively, then the value of the determinant  $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$  is

(a)  $\Delta$  (b)  $\Delta^2$  (c)  $\Delta^3$  (d) 0

**Solution:** (b) We know that  $\Delta \Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} \Sigma a_1 A_1 & 0 & 0 \\ 0 & \Sigma a_2 A_2 & 0 \\ 0 & 0 & \Sigma a_3 A_3 \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$

$\Rightarrow \Delta' = \Delta^2$

**Trick :** According to property of cofactors  $\Delta' = \Delta^{n-1} = \Delta^2$  ( $\because$  Hence  $n=3$ )

**Example: 15** If the value of a third order determinant is 11, then the value of the square of the determinant formed by the cofactors will be

(a) 11 (b) 121 (c) 1331 (d) 14641

**Solution:** (d)  $\Delta' = \Delta^{n-1} = \Delta^{3-1} = \Delta^2 = (11)^2 = 121$ . But we have to find the value of the square of the determinant, so required value is  $(121)^2 = 14641$ .

### 8.1.6 Product of two Determinants

Let the two determinants of third order be,

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}. \text{ Let } D \text{ be their product.}$$



Then,  $\sum_{r=1}^n \Delta_r = \begin{vmatrix} \sum_{r=1}^n f(r) & a & l \\ \sum_{r=1}^n g(r) & b & m \\ \sum_{r=1}^n h(r) & c & n \end{vmatrix}$ . Here function of  $r$  can be the elements of only one row or

one column.

**Example: 18** If  $D_p = \begin{vmatrix} p & 15 & 8 \\ p^2 & 35 & 9 \\ p^3 & 25 & 10 \end{vmatrix}$ , then  $D_1 + D_2 + D_3 + D_4 + D_5 =$

- (a) 0                                      (b) 25                                      (c) 625                                      (d) None of these

**Solution:** (d)  $\because D_1 = \begin{vmatrix} 1 & 15 & 8 \\ 1 & 35 & 9 \\ 1 & 25 & 10 \end{vmatrix}$ ,  $D_2 = \begin{vmatrix} 2 & 15 & 8 \\ 4 & 35 & 9 \\ 8 & 25 & 10 \end{vmatrix}$ ,  $D_3 = \begin{vmatrix} 3 & 15 & 8 \\ 9 & 35 & 9 \\ 27 & 25 & 10 \end{vmatrix}$ ,  $D_4 = \begin{vmatrix} 4 & 15 & 8 \\ 16 & 35 & 9 \\ 64 & 25 & 10 \end{vmatrix}$ ,  $D_5 = \begin{vmatrix} 5 & 15 & 8 \\ 25 & 35 & 9 \\ 125 & 25 & 10 \end{vmatrix}$

$$\Rightarrow D_1 + D_2 + D_3 + D_4 + D_5 = \begin{vmatrix} 15 & 75 & 40 \\ 55 & 175 & 45 \\ 225 & 125 & 50 \end{vmatrix}$$

$$= 15(3125) - 75(-32500) + 40(-32500) = 46875 + 553125 - 1300000 = -700000$$

**Example: 19** The value of  $\sum_{n=1}^N U_n$ , if  $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N \end{vmatrix}$  is

- (a) 0                                      (b) 1                                      (c) -1                                      (d) None of these

**Solution:** (a)  $\sum_{n=1}^N U_n = \begin{vmatrix} \frac{N(N+1)}{2} & 1 & 5 \\ \frac{N(N+1)(2N+1)}{6} & 2N+1 & 2N+1 \\ \left\{ \frac{N(N+1)}{2} \right\}^2 & 3N^2 & 3N \end{vmatrix} = \frac{N(N+1)}{12} \begin{vmatrix} 6 & 1 & 5 \\ 4N+2 & 2N+1 & 2N+1 \\ 3N(N+1) & 3N^2 & 3N \end{vmatrix}$

Applying  $C_3 \rightarrow C_3 + C_2 = \frac{N(N+1)}{12} \begin{vmatrix} 6 & 1 & 6 \\ 4N+2 & 2N+1 & 4N+2 \\ 3N(N+1) & 3N^2 & 3N(N+1) \end{vmatrix} = 0$  [ $\because C_1$  and  $C_3$  are

identical]

### 8.1.8 Differentiation and Integration of Determinants

(1) **Differentiation of a determinant :** (i) Let  $\Delta(x)$  be a determinant of order two. If we write  $\Delta(x) = \begin{vmatrix} C_1 & C_2 \end{vmatrix}$ , where  $C_1$  and  $C_2$  denote the 1<sup>st</sup> and 2<sup>nd</sup> columns, then

$$\Delta'(x) = \begin{vmatrix} C'_1 & C_2 \end{vmatrix} + \begin{vmatrix} C_1 & C'_2 \end{vmatrix}$$

where  $C'_i$  denotes the column which contains the derivative of all the functions in the  $i^{\text{th}}$  column  $C_i$ .

In a similar fashion, if we write  $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \end{vmatrix}$ , then  $\Delta'(x) = \begin{vmatrix} R'_1 \\ R_2 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \end{vmatrix}$

(ii) Let  $\Delta(x)$  be a determinant of order three. If we write  $\Delta(x) = \begin{vmatrix} C_1 & C_2 & C_3 \end{vmatrix}$ , then

$$\Delta'(x) = |C'_1 \ C_2 \ C_3| + |C_1 \ C'_2 \ C_3| + |C_1 \ C_2 \ C'_3|$$

and similarly if we consider  $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$ , then  $\Delta'(x) = \begin{vmatrix} R'_1 \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R'_3 \end{vmatrix}$

(iii) If only one row (or column) consists functions of  $x$  and other rows (or columns) are constant, viz.

Let  $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ ,

then  $\Delta'(x) = \begin{vmatrix} f'_1(x) & f'_2(x) & f'_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$  and in general  $\Delta^n(x) = \begin{vmatrix} f_1^n(x) & f_2^n(x) & f_3^n(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

where  $n$  is any positive integer and  $f^n(x)$  denotes the  $n^{\text{th}}$  derivative of  $f(x)$ .

**Example: 20** If  $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$  and  $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$  are the given determinants, then

[MNR 1986; Kurukshetra CEE 1998; UPSEAT 2000]

- (a)  $\Delta_1 = 3(\Delta_2)^2$       (b)  $\frac{d}{dx}(\Delta_1) = 3\Delta_2$       (c)  $\frac{d}{dx}(\Delta_1) = 2(\Delta_2)^2$       (d)  $\Delta_1 = 3\Delta_2^{3/2}$

**Solution:** (b)  $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix} = x^3 - 3abx \Rightarrow \frac{d}{dx}(\Delta_1) = 3(x^2 - ab)$  and  $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix} = x^2 - ab$

$\therefore \frac{d}{dx}(\Delta_1) = 3(x^2 - ab) = 3\Delta_2$

**Example: 21** If  $y = \sin mx$ , then the value of the determinant  $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ , where  $y_n = \frac{d^n y}{dx^n}$  is

- (a)  $m^9$       (b)  $m^2$       (c)  $m^3$       (d) None of these

**Solution:** (d)  $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix} = \begin{vmatrix} \sin mx & m \cos mx & -m^2 \sin mx \\ -m^3 \cos mx & m^4 \sin mx & m^5 \cos mx \\ -m^6 \sin mx & -m^7 \cos mx & m^8 \sin mx \end{vmatrix}$

Taking  $-m^6$  common from  $R_3$ ,  $R_1$  and  $R_3$  becomes identical. Hence the value of determinant is zero.

(2) **Integration of a determinant :** Let  $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix}$ , where  $a, b, c, l, m$  and  $n$  are

constants.

$$\Rightarrow \int_a^b \Delta(x) dx = \begin{vmatrix} \int_a^b f(x) dx & \int_a^b g(x) dx & \int_a^b h(x) dx \\ a & b & c \\ l & m & n \end{vmatrix}$$

**Note:** □ If the elements of more than one column or rows are functions of  $x$  then the integration can be done only after evaluation/expansion of the determinant.

**Example: 22** If  $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1 - \cos x \\ 1 + \sin x & \cos x & 1 + \sin x - \cos x \\ \sin x & \sin x & 1 \end{vmatrix}$ , then  $\int_0^{\pi/2} \Delta(x) dx$  is equal to

- (a)  $1/4$                       (b)  $1/2$                       (c)  $0$                       (d)  $-1/2$

**Solution:** (d) Applying  $C_3 \rightarrow C_3 + C_2 - C_1$

$$\Delta(x) = \begin{vmatrix} 1 & \cos x & 0 \\ 1 + \sin x & \cos x & 0 \\ \sin x & \sin x & 1 \end{vmatrix} = \cos x - \cos x(1 + \sin x) = -\sin x \cos x$$

$$\therefore \int_0^{\pi/2} \Delta(x) dx = -\frac{1}{2} \int_0^{\pi/2} \sin 2x dx = -\frac{1}{2} \left[ -\frac{\cos 2x}{2} \right]_0^{\pi/2} = \frac{1}{4} (\cos \pi - \cos 0) = -\frac{1}{2}$$

### 8.1.9 Application of Determinants in solving a system of Linear Equations

Consider a system of simultaneous linear equations is given by

$$\left. \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \right\}$$

.....(i)

A set of values of the variables  $x, y, z$  which simultaneously satisfy these three equations is called a solution. A system of linear equations may have a unique solution or many solutions, or no solution at all, if it has a solution (whether unique or not) the system is said to be consistent. If it has no solution, it is called an inconsistent system.

If  $d_1 = d_2 = d_3 = 0$  in (i) then the system of equations is said to be a homogeneous system. Otherwise it is called a non-homogeneous system of equations.

**Theorem 1 :** (Cramer's rule) The solution of the system of simultaneous linear equations

$$a_1x + b_1y = c_1 \quad \text{.....(i)} \quad \text{and} \quad a_2x + b_2y = c_2 \quad \text{.....(ii)}$$

is given by  $x = \frac{D_1}{D}, y = \frac{D_2}{D}$ , where  $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ ,  $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$  and  $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ , provided

that  $D \neq 0$

**Note:** □ Here  $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$  is the determinant of the coefficient matrix  $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ .

The determinant  $D_1$  is obtained by replacing first column in  $D$  by the column of the right hand side

of the given equations. The determinant  $D_2$  is obtained by replacing the second column in  $D$  by the right most column in the given system of equations.

**(1) Solution of system of linear equations in three variables by Cramer's rule :**

**Theorem 2 :** (Cramer's Rule) The solution of the system of linear equations

$$a_1x + b_1y + c_1z = d_1 \quad \text{.....(i)}$$

$$a_2x + b_2y + c_2z = d_2 \quad \text{.....(ii)}$$

$$a_3x + b_3y + c_3z = d_3 \quad \text{.....(iii)}$$



(iii) If  $D=0$  and at least one of the determinants  $D_1, D_2, D_3$  is non-zero, then given of equations is inconsistent.

(3) **Algorithm for solving a system of simultaneous linear equations by Cramer's rule (Determinant method)**

**Step 1 :** Obtain  $D, D_1, D_2$  and  $D_3$

**Step 2 :** Find the value of  $D$ . If  $D \neq 0$ , then the system of the equations is consistent has a unique solution. To find the solution, obtain the values of  $D_1, D_2$  and  $D_3$ . The solutions is given by  $x = \frac{D_1}{D}, y = \frac{D_2}{D}$  and  $z = \frac{D_3}{D}$ . If  $D=0$  go to step 3.

**Step 3 :** Find the values of  $D_1, D_2, D_3$ . If at least one of these determinants is non-zero, then the system is inconsistent. If  $D_1 = D_2 = D_3 = 0$ , then go to step 4

**Step 4 :** Take any two equations out of three given equations and shift one of the variables, say  $z$  on the right hand side to obtain two equations in  $x, y$ . Solve these two equations by Cramer's rule to obtain  $x, y$ , in terms of  $z$ .

**Note:**  $\square$  The system of following homogeneous equations  $a_1x + b_1y + c_1z = 0$ ,  $a_2x + b_2y + c_2z = 0$ ,  $a_3x + b_3y + c_3z = 0$  is always consistent.

If  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ , then this system has the unique solution  $x = y = z = 0$  known

as **trivial solution**. But if  $\Delta = 0$ , then this system has an infinite number of solutions. Hence for non-trivial solution  $\Delta = 0$ .

**Example: 23** If the system of linear equations  $x + 2ay + az = 0$ ,  $x + 3by + bz = 0$ ,  $x + 4cy + cz = 0$  has a non-zero solution, then  $a, b, c$

[AIEEE 2003]

- (a) Are in A.P.                      (b) Are in G.P.                      (c) Are in H.P.                      (d) Satisfy  $a + 2b + 3c = 0$

**Solution:** (c) System of linear equations has a non-zero solution, then  $\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$

Applying  $C_2 \rightarrow C_2 - 2C_3$ ;  $\begin{vmatrix} 1 & 0 & a \\ 1 & b & b \\ 1 & 2c & c \end{vmatrix} = 0$

Applying  $R_3 \rightarrow R_3 - R_2$  and  $R_2 \rightarrow R_2 - R_1$

$\begin{vmatrix} 1 & 0 & a \\ 0 & b & b-a \\ 0 & 2c-b & c-b \end{vmatrix} = 0 \Rightarrow b(c-b) - (b-a)(2c-b) = 0$ . On simplification  $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$ ;  $\therefore a, b, c$  are in H.P.

**Example: 24** If the system of equations  $x + ay = 0$ ,  $az + y = 0$  and  $ax + z = 0$  has infinite solutions, then the value of  $a$  is

[IIT Screening 2003]

- (a) -1                                      (b) 1                                      (c) 0                                      (d) No real values



**Solution:** (a)  $\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1 + a(a)^2 = 0 \Rightarrow a^3 = -1 \Rightarrow a = -1$

**Example: 25** If the system of equations  $ax + y + z = 0$ ,  $x + by + z = 0$  and  $x + y + cz = 0$ , where  $a, b, c \neq 1$  has a non-trivial solution, then the value of  $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$  is

- (a) -1                                      (b) 0                                      (c) 1                                      (d) None of these

**Solution:** (c) As the system of the equations has a non-trivial solution  $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

Applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$

$$\begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 1-a & 0 & c-1 \end{vmatrix} = 0 \Rightarrow a(b-1)(c-1) - 1(1-a)(c-1) - 1(1-a)(b-1) = 0$$

$$\Rightarrow \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0 \Rightarrow \frac{1}{1-a} - 1 + \frac{1}{1-b} + \frac{1}{1-c} = 0 \Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

**Example: 26** If the system of equations  $x + 2y - 3z = 1$ ,  $(k + 3)z = 3$ ,  $(2k + 1)x + z = 0$  is inconsistent, then the value of  $k$  is

[Roorkee 2000]

- (a) -3                                      (b)  $\frac{1}{2}$                                       (c) 0                                      (d) 2

**Solution:** (a) For the equations to be inconsistent  $D = 0$

$$\therefore D = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & k+3 \\ 2k+1 & 0 & 1 \end{vmatrix} = 0 \Rightarrow k = -3 \text{ and } D_1 = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0, \text{ Hence system is inconsistent for } k = -3.$$

**Example: 27.** The equations  $x + y + z = 6$ ,  $x + 2y + 3z = 10$ ,  $x + 2y + mz = n$  give infinite number of values of the triplet  $(x, y, z)$  if

□

- (a)  $m = 3, n \in R$                       (b)  $m = 3, n \neq 10$                       (c)  $m = 3, n = 10$                       (d) None of these

**Solution:** (c) Each of the first three options contains  $m = 3$ . When  $m = 3$ , the last two equations become  $x + 2y + 3z = 10$  and  $x + 2y + 3z = n$ .

Obviously, when  $n = 10$  these equations become the same. So we are left with only two independent equations to find the values of the three unknowns.

Consequently, there will be infinite solutions.

**Example: 28** The value of  $\lambda$  for which the system of equations  $2x - y - z = 12$ ,  $x - 2y + z = -4$ ,  $x + y + \lambda z = 4$  has no solution is

[IIT Screening 2004]

- (a) 3                                      (b) -3                                      (c) 2                                      (d) -2

**Solution:** (d)  $D = \begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = -3\lambda - 6$ . For no solution the necessary condition is  $-3\lambda - 6 = 0 \Rightarrow \lambda = -2$ . It can be seen

that for  $\lambda = -2$ , there is no solution for the given system of equations.

### 8.1.10 Application of Determinants in Co-ordinate Geometry

(1) Area of triangle whose vertices are  $(x_r, y_r)$ ;  $r = 1, 2, 3$  is

$$\Delta = \frac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(2) If  $a_r x + b_r y + c_r = 0$ , ( $r = 1, 2, 3$ ) are the sides of a triangle, then the area of the triangle is given by

$$\Delta = \frac{1}{2C_1 C_2 C_3} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2, \text{ where } C_1 = a_2 b_3 - a_3 b_2, C_2 = a_3 b_1 - a_1 b_3, C_3 = a_1 b_2 - a_2 b_1 \text{ are the}$$

cofactors of the elements  $c_1, c_2, c_3$  respectively in the determinant  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ .

(3) The equation of a straight line passing through two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(4) If three lines  $a_r x + b_r y + c_r = 0$ ; ( $r = 1, 2, 3$ ) are concurrent if  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(5) If  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents a pair of straight lines then

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(6) The equation of circle through three non-collinear points  $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$  is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

**Example: 29** The three lines  $ax + by + c = 0$ ,  $bx + cy + a = 0$ ,  $cx + ay + b = 0$  are concurrent only when

(a)  $a + b + c = 0$

(b)  $a^2 + b^2 + c^2 = ab + bc + ca$

(c)  $a^3 + b^3 + c^3 = ab + bc + ca$

(d) None of these

**Solution:** (a, b) Three lines are concurrent if  $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$  or,  $3abc - a^3 - b^3 - c^3 = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

Also,  $a^3 + b^3 + c^3 - 3abc = 0 \Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow (a + b + c) = 0$  or  $a^2 + b^2 + c^2 = ab + bc + ca$ .

### 8.1.11 Some Special Determinants

(1) **Symmetric determinant** : A determinant is called symmetric determinant if for its every

element  $a_{ij} = a_{ji} \forall i, j$  e.g., 
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(2) **Skew-symmetric determinant** : A determinant is called skew symmetric determinant if for

its every element  $a_{ij} = -a_{ji} \forall i, j$  e.g., 
$$\begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{vmatrix}$$

**Note** :  $\square$  Every diagonal element of a skew symmetric determinant is always zero.

$\square$  The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

For example (i) 
$$\begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = -c(0 - ab) - b(ac - 0) = abc - abc = 0$$

(ii) 
$$\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = 0 + a^2 = a^2 \text{ (Perfect square)}$$
      (iii) 
$$\begin{vmatrix} 0 & a-b & e-f \\ b-a & 0 & l-m \\ f-e & m-l & 0 \end{vmatrix} = 0$$

(3) **Cyclic order** : If elements of the rows (or columns) are in cyclic order.

i.e. (i) 
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(ii) 
$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

(iii) 
$$\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$$

(iv) 
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$
      (v) 
$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

**Note** :  $\square$  These results direct applicable in lengthy questions (As behavior of standard results)

**Example: 30** 
$$\Delta = \begin{vmatrix} 0 & i-100 & i-500 \\ 100-i & 0 & 1000-i \\ 500-i & i-1000 & 0 \end{vmatrix}$$
 is equal to

- (a) 100                                      (b) 500                                      (c) 1000                                      (d) 0

**Solution:** (d) This determinant of skew symmetric of odd order, hence is equal to 0.

**Example: 31**  $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} =$  [Pb. CET 1997; DCE 2002]

- (a)  $a^2 + b^2 + c^2$       (b)  $(a+b)(b+c)(c+a)$       (c)  $(a-b)(b-c)(c-a)$       (d) None of these

**Solution: (c)**  $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Applying  $R_1 \rightarrow R_1 - R_2$  &  $R_2 \rightarrow R_2 - R_3$

$$= \begin{vmatrix} 0 & a-b & a^2 - b^2 \\ 0 & b-c & b^2 - c^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix};$$

Applying  $R_1 \rightarrow R_1 - R_2$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & (a-c) \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(a-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(a-c)(-1) = (a-b)(b-c)(c-a)$$

**Example: 32** If  $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$  Where  $a, b, c$  are all different, then the determinant

$\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$  vanishes when

- (a)  $a+b+c=0$       (b)  $x = \frac{1}{3}(a+b+c)$       (c)  $x = \frac{1}{2}(a+b+c)$       (d)  $x = a+b+c$

**Solution: (b)**  $\because \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$  ..... (i)

Now,  $\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{(x-a)(x-b)(x-c)} \begin{vmatrix} (x-a) & (x-b) & (x-c) \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x-a)(x-b)(x-c) & (x-a)(x-b)(x-c) & (x-a)(x-b)(x-c) \end{vmatrix} = 0$$

Applying  $C_1 \rightarrow C_1(x-a), C_2 \rightarrow C_2(x-b), C_3 \rightarrow C_3(x-c)$

$$\begin{vmatrix} (x-a) & (x-b) & (x-c) \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$[(x-a)-(x-b)][(x-b)-(x-c)][(x-c)-(x-a)](x-a+x-b+x-c) = 0$$

$$(b-a)(c-b)(a-c)[3x-(a+b+c)] = 0 \text{ or } x = \frac{1}{3}(a+b+c) \quad [:\because a \neq b \neq c]$$

**Example: 33** If  $\alpha, \beta$  and  $\gamma$  are the roots of the equations  $x^3 + px + q = 0$  then value of the determinant  $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$  is

[AMU 1990]

- (a)  $p$       (b)  $q$       (c)  $p^2 - 2q$       (d)  $0$

**Solution: (d)** Since  $\alpha, \beta, \gamma$  are the roots of  $x^3 + px + q = 0, \therefore \alpha + \beta + \gamma = 0$

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + R_2 + R_3$ , We get,  $\begin{vmatrix} \alpha + \beta + \gamma & \alpha + \beta + \gamma & \alpha + \beta + \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0$

- Example: 34** If  $\Delta = \begin{vmatrix} x+1 & x^2+2 & x(x+1) \\ x(x+1) & x+1 & x(x^2+2) \\ x^2+2 & x(x+1) & x+1 \end{vmatrix} = p_0x^6 + p_1x^5 + p_2x^4 + p_3x^3 + p_4x^2 + p_5x + p_6$ , then  $(p_5, p_6) =$
- (a)  $(-3, -9)$                       (b)  $(-5, -9)$                       (c)  $(-3, -5)$                       (d)  $(3, -9)$

**Solution:** (b) Putting  $x = 0$  in both sides, we get,  $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = p_6 \Rightarrow p_6 = 9$  by expansion.

$p_5$  is the coefficient of  $x$  or constant term in the differentiation of determinant.

Differentiate both sides,

$$\begin{vmatrix} 1 & 2x & 2x+1 \\ x(x+1) & x+1 & x(x^2+2) \\ x^2+2 & x(x+1) & x+1 \end{vmatrix} + \begin{vmatrix} x+1 & x^2+2 & x(x+1) \\ 2x+1 & 1 & 3x^2+2 \\ x^2+2 & x(x+1) & x+1 \end{vmatrix} + \begin{vmatrix} x+1 & x^2+2 & x(x+1) \\ x(x+1) & x+1 & x(x^2+2) \\ 2x & 2x+1 & 1 \end{vmatrix} \\ = 6p_0x^5 + 5p_1x^4 + 4p_2x^3 + 3p_3x^2 + 2p_4x + p_5$$

Putting  $x = 0$  both sides, we get  $p_5 = -5$ ;  $\therefore (p_5, p_6) = (-5, 9)$ .

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