# 4.6 Lagrange's Mean Value Theorem

## 4.6.1 Definition

If a function $f(x)$,

(1) Is continuous in the closed interval $[a, b]$ and

(2) Is differentiable in the open interval $(a, b)$

Then there is atleast one value $c \in (a,b)$, such that; $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

## 4.6.2 Analytical Interpretation

**First form:** Consider the function, $\phi(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}x$

Since, $f(x)$ is continuous in $[a, b]$

$\therefore$ $\phi(x)$ is also continuous in $[a,b]$

since, $f'(x)$ exists in $(a, b)$ hence $\phi'(x)$ also exists in $(a,b)$ and $\phi'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$ .....(i)

Clearly, $\phi(x)$ satisfies all the condition of Rolle's theorem

$\therefore$ There is atleast one value of $c$ of $x$ between $a$ and $b$ such that $\phi'(c) = 0$ substituting $x = c$ in (i) we get,

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$ which proves the theorem.

**Second form:** If we write $b = a + h$ then $a < c < b, c = a + \theta h$ where $0 < \theta < 1$

Thus, the mean value theorem can be stated as follows:

If (i) $f(x)$ is continuous in closed interval $[a, a+h]$

(ii) $f'(x)$ exists in the open interval $(a, a+h)$ then there exists at least one number $\theta (0 < \theta < 1)$

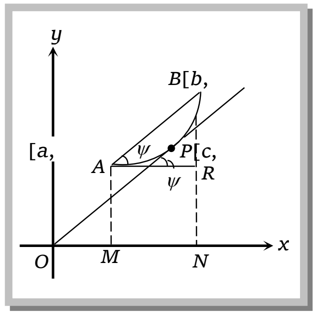
Such that $f(a+h) = f(a) + hf'(a+\theta h)$.

## 4.6.3 Geometrical Interpretation

Let $f(x)$ be a function defined on $[a, b]$ and let $APB$ be the curve represented by $y = f(x)$. Then co-ordinates of $A$ and $B$ are $(a, f(a))$ and $(b, f(b))$ respectively. Suppose the chord $AB$ makes an angle $\psi$ with the axis of $x$. Then from the triangle $ARB$, we have

$$\tan\psi = \frac{BR}{AR} \Rightarrow \tan\psi = \frac{f(b)-f(a)}{b-a}$$

By Lagrange's Mean value theorem, we have, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ $\therefore \tan\psi = f'(c)$




$\Rightarrow$ slope of the chord $AB$ = slope of the tangent at $(c, f(c))$

**Example: 1** In the mean-value theorem $\dfrac{f(b)-f(a)}{b-a} = f'(c)$, if $a = 0$, $b = \dfrac{1}{2}$ and $f(x) = x(x-1)(x-2)$, the value of $c$ is **[MP PET 20**

(a) $1 - \dfrac{\sqrt{15}}{6}$ (b) $1 + \sqrt{15}$ (c) $1 - \dfrac{\sqrt{21}}{6}$ (d) $1 + \sqrt{21}$

**Solution:** (c) From mean value theorem $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$a = 0, f(a) = 0 \Rightarrow b = \dfrac{1}{2}, f(b) = \dfrac{3}{8}$

$f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$,

$f'(c) = (c-1)(c-2) + c(c-2) + c(c-1) = c^2 - 3c + 2 + c^2 - 2c + c^2 - c$, $f'(c) = 3c^2 - 6c + 2$

According to mean value theorem

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 3c^2 - 6c + 2 = \frac{\left(\frac{3}{8}\right) - 0}{\left(\frac{1}{2}\right) - 0} = \frac{3}{4} \Rightarrow 3c^2 - 6c + \frac{5}{4} = 0$$

$$c = \frac{6 \pm \sqrt{36-15}}{2 \times 3} = \frac{6 \pm \sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{6}.$$

**Example: 2** From mean value theorem $f(b) - f(a) = (b-a) f'(x_1), a < x_1 < b$ if $f(x) = \dfrac{1}{x}$ then $x_1$

(a) $\sqrt{ab}$ (b) $\dfrac{2ab}{a+b}$ (c) $\dfrac{a+b}{2}$ (d) $\dfrac{b-a}{b+a}$

**Solution:** (a) $f'(x_1) = \dfrac{-1}{x_1^2}$, $\therefore \dfrac{-1}{x_1^2} = \dfrac{\frac{1}{b} - \frac{1}{a}}{b-a} = -\dfrac{1}{ab} \Rightarrow x_1 = \sqrt{ab}$.

**Example: 3** The abscissae of the points of the curve $y = x^3$ in the interval [–2, 2], where the slope of the tangent can be obtained by mean value theorem for the interval [– 2, 2] are

(a) $\pm\dfrac{2}{\sqrt{3}}$ (b) $\pm\dfrac{\sqrt{3}}{2}$ (c) $\pm\sqrt{3}$ (d) 0

**Solution:** (a) Given that equation of curve $y = x^3 = f(x)$

So $f(2)=8$ and $f(-2)=-8$

Now $f'(x)=3x^2 \Rightarrow f'(x)=\dfrac{f(2)-f(-2)}{2-(-2)} \Rightarrow \dfrac{8-(-8)}{4}=3x^2; \therefore x=\pm\dfrac{2}{\sqrt{3}}$.

# Assignment

## Lagrange's Mean Value Theorem

### Basic Level

**1.** If from mean value theorem, $f'(x_1) = \frac{f(b) - f(a)}{b - a}$, then **[MP PET 1999]**

(a) $a < x_1 \le b$ (b) $a \le x_1 < b$ (c) $a < x_1 < b$ (d) $a \le x_1 \le b$

**2.** For the function $x + \frac{1}{x}, x \in [1, 3]$, the value of $c$ for the mean value theorem is **[MP PET 1997]**

(a) 1 (b) $\sqrt{3}$ (c) 2 (d) None of these

**3.** For the function $f(x) = e^x, a = 0, b = 1$, the value of c in mean value theorem will be **[DCE 2002]**

(a) log $x$ (b) log $(e - 1)$ (c) 0 (d) 1

### Advance Level

**4.** If the function $f(x) = x^3 - 6ax^2 + 5x$ satisfies the conditions of Lagrange's mean value theorem for the interval [1, 2] and the tangent to the curve $y = f(x)$ at $x = \frac{7}{4}$ is parallel to the chord that joins the points of intersection of the curve with the ordinates $x = 1$ and $x = 2$. Then the value of $a$ is

(a) $35/16$ (b) $35/48$ (c) $7/16$ (d) $5/16$

**5.** If $f(x) = \cos x, 0 \le x \le \frac{\pi}{2}$, then the real number 'c' of the mean value theorem is **[MP PET 1994]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\sin^{-1}\left(\frac{2}{\pi}\right)$ (d) $\cos^{-1}\left(\frac{2}{\pi}\right)$

**6.** Let $f(x)$ satisfy all the conditions of mean value theorem in [0, 2]. If $f(0) = 0$ and $|f'(x)| \le \frac{1}{2}$ for all $x$ in [0, 2], then

(a) $f(x) \le 2$ (b) $|f(x)| \le 1$

(c) $f(x) = 2x$ (d) $f(x) = 3$ for at least one $x$ in [0, 2]

**7.** The function $f(x)=(x-3)^2$ satisfies all the conditions of mean value theorem in [3, 4]. A point on $y=(x-3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1) is

(a) $\left(\frac{7}{2},\frac{1}{2}\right)$ (b) $\left(\frac{7}{2},\frac{1}{4}\right)$ (c) $(1,4)$ (d) (4, 1)

**8.** Let $f(x)$ and $g(x)$ are defined and differentiable for $x \geq x_0$ and $f(x_0)=g(x_0), f'(x)>g'(x)$ for $x>x_0$, then

(a) $f(x)<g(x)$ for some $x>x_0$

(b) $f(x)=g(x)$ for some $x>x_0$

(c) $f(x)>g(x)$ for all $x>x_0$

(d) None of these

**9.** Let $f$ be differentiable for all $x$. If $f(1)=-2$ and $f'(x)\geq 2$ for all $x\in[1,6]$ then

(a) $f(6)<8$ (b) $f(6)\geq 8$ (c) $f(6)\geq 5$ (d) $f(6)\leq 5$

**10.** The value of $c$ in Lagrange's theorem for the function $|x|$ in the interval [– 1, 1] is

(a) 0 (b) 1/2 (c) –1/2 (d) Non-existent in the interval

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# Answer Sheet

## *Assignment (Basic and Advance Level)*

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| c | b | b | b | c | b | b | c | b | d |